SOME INVESTIGATIONS INTO CAUSAL VISCOUS COSMOLOGICAL MODELS WITH VARIABLE LAMBDA

by

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DEDICATION

This research is dedicated to my father S.A. Nkosi, who has been a pillar of strength through out my life, my wife Letta and all other family members who spent long hours of isolation because of this research.

ABSTRACT

In this thesis we investigate the evolution of viscous Friedmann-Lemaitre-Robertson-Walker models with a variable cosmological term. The numerical solutions are obtained and represented graphically. Each graph depends on the value of gamma. The present values of the deceleration parameter and density parameter were obtained by using Eckart theory and the truncated theory and given in the tabular form.

Power law solutions for the Hubble parameter are shown to exist and we give the values of all the other cosmological variables. The behaviour of the temperature depends on the initial conditions. Furthermore, the equations are also transformed into a plane autonomous system by using dimensionless variables and a dimensionless equation of state, and the qualitative behaviour of the system is investigated. The sets of equilibrium points are determined and their behaviour discussed. The exact value of the Hubble parameter, cosmological term, bulk viscosity pressure and the energy density are obtained and discussed.

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Chapter I

Introduction

1. Introduction

Cosmology is regarded as the study of the large scale structure and evolution of the universe. According to the cosmological principle, the entire smoothed-out universe at a fixed time is homogeneous and isotropic. In studying the universe, local irregularities are replaced with an average matter tensor which is taken to be the same at all places. In cosmology, one needs to consider questions such as the expansion of the universe, the structure of the microwave background radiation, how the universe may have been created and how the structures like galaxies were formed.

The big-bang cosmology proposes that the entire universe was created in a big explosion of space-time in the past. The early universe was very hot and radiation filled, and it is expanding as shown by the redshifts of the distant galaxies. The universe is believed to be flat to a remarkable degree of accuracy, and on a large scale, the universe appears to be homogeneous and isotropic.

The history of the universe since 1917 shows great discoveries and increasing understanding of the universe especially after 1965. The discoveries are classified according to three periods. The initial period (1917-1930) was where quantitative cosmological models were being developed, but their significance and particularly the idea of an expanding universe was not understood. The second period (1930-1945) was the time for consolidation, when the dynamical and geographical aspects of the idea of an expanding universe were extensively explored and developed. The third period (1945-1960) was the time when the mathematical development and investigations of the astrophysical aspects of the expanding universe laid the foundations for the dramatic developments of the following years.

The great theory that leads to the study of cosmology is Einstein's theory of general relativity. In 1917 when Einstein proposed his static universe model, there was insufficient evidence to show that the universe might be expanding because the distances of galaxies was not accurately known. To introduce a static solution, Einstein introduced into the field equations the cosmological term Λ which made a static solution possible. This cosmological term is responsible for the purpose of making a quasi-static distribution of matter. The cosmological term allowed the first quantitative cosmological model to be uniform in space and time, without evolution taking place. The cosmological term Λ as a function of time has also been considered by several authors in different contexts.

The role of viscosity and dissipative mechanisms in cosmology have been studied by many authors. It has been shown that the presence of bulk viscosity leads to an inflationary negative energy field in an expanding universe. During the evolution of the universe there are number of circumstances in which bulk viscosity could have arisen. Some of these processes may be understood by conventional physics and other processes involve the partly speculative physics of the early universe, for instance, (1) when neutrinos decouple from the cosmic fluid, when the temperature falls below the threshold interaction that keeps the neutrinos in thermal equilibrium, (2) when photons decouple from matter during the recombination era, (3) at the time for formation of galaxies, (4) during a collapse of a radiative star to a black hole, when neutrino emission is responsible for dissipative heat flow and viscosity.

The first noted attempts to build dissipative cosmological models were based on the theory of Eckart, and Landau and Lifshitz. Eckart theory plays a crucial role in cosmology. The problem with this model is that it considers only first order deviations from equilibrium. Hiscock and Lindblom [1] have pointed out that these theories are not satisfactory relativistic theories because they are non-causal and their equilibrium states are unstable. Also perturbations do not have a well-posed initial value problem. Maartens [40] indicated that a necessary requirement to prevent non-causal and unstable behaviour is to include the second order term.

Early attempts to build up causally well-behaved viscous fluid models were made by Israel [2] and subsequently developed by Israel and Stewart [3], Pavon *et al* [4]; and Jou *et al* [5]. This theory is shown to be causal and stable under reasonable conditions, and is called a second order theory.

The effect of the second order term on the evolution of the models was discussed partly by Oliviera and Salim [19], Zakari and Jou [11], and Chimento and Jakubi [12]. For this model also problems arise because it omits certain divergence terms in the causal evolution equation as indicated by Hiscock and Salmonson [37]. The truncated equation is formed and it can lead to pathological behaviour of the temperature [51]. Maartens [40] argues that the truncated theory agrees with the full theory if one uses, instead of local equilibrium variables, a generalized temperature and pressure.

The qualitative behaviour of flat Friedmann-Lemaitre Robertson Walker models was determined by various authors including Coley *et al* [36] and Mendez and Triginer [46] and some exact solutions were obtained.

In this thesis, we investigate some cosmological models with causal bulk viscosity and a variable cosmological term. Chapter 1 of this work begins with the investigation of numerical solutions in both Eckart and truncated theories for $m = \frac{1}{2}$, $m = 1$ and $m = \frac{3}{2}$, where m is the power of the energy density in the equation of state for the coefficient of bulk viscosity, and depends on the relativistic fluid under consideration. The equation of state for the bulk viscosity is given by $\xi = \alpha\rho^m$, where α is a constant. If $m = 1$, the relationship $\xi = \alpha\rho^m$ corresponds to a radiative fluid (Jou and Pavon [5]) and for $m = \frac{3}{2}$, it corresponds to a string dominated universe in the sense of Turok [6]. The solutions are represented graphically.

Most of work done in cosmological solutions with viscosity are done without a variable cosmological term. However, since the cosmological term plays a major role, in this work we want to show the importance of a variable Λ term, through different methods of analysis. This is the motivation for the present work.

In chapter 2 we determine some exact solutions of flat Friedmann-Lemaitre-Robertson-Walker models using the full (non-truncated) theory for $m = \frac{1}{2}$, and the cosmological term Λ is directly proportional to the square of the Hubble parameter H. The equation is $\Lambda = 3\beta H^2$, where β is a constant.

In chapter 3, we determine the qualitative behaviour of the same equation using the same value of Λ for $n = 0, n = 1$ and $n = 2$, where n is the power of the dimensionless density parameter in the equation of the expansion rate $(\frac{\tau^{-1}}{H} = bz^n)$. We finally obtain the exact dynamic behaviour of the parameters H, Λ, R, ρ and Π

2. Basic definitions, theories and equations

2.1. First order theories

This theory depends on the stress-energy momentum tensor of matter T^{ab}, which splits into :

$$T^{ab} = (\rho + p_{eff})u^a u^b + p_{eff} g^{ab} + q^a u^b + q^b u^a + \Pi^{ab} - \frac{\Lambda c^4}{8\pi G} g^{ab}$$

and the number current vector N^a, which obeys the conservation laws:

$$T^{ab}_{;b} = 0$$

and

$$N^a_{;a} = 0$$

where ρ is the energy density, p_{eff} is the effective pressure, q^a is the heat flow vector, Π^{ab} is the shear tensor, u^a is the four velocity and Λ is the cosmological term. The energy momentum tensor is the source of the gravitational field defined by :

$$T_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$$

where R_{ab} is the Ricci tensor and R is the Ricci scalar. For

$$p_{eff} = q^a = \Pi = \Lambda = 0$$

we have perfect fluids. The effective pressure is given by $p_{eff} = p - 3\xi H$, where p is the equilibrium pressure.

2.2. Second order theories

The second order theory as derived by Israel and Stewart and is actually a collection of theories each of which defines the four velocity vector u^a in terms of fundamental tensors, in a different way. In the first case u^a is defined to be parallel to the particle number N^a, and in the second case one can define u^a to be a time-like eigenvector of T^{ab}. In these theories also the fundamental variable for describing the relativistic fluid is the tensor T^{ab} and vector N^a, where N^a is given by

$$N^a = nu^a$$

where n is the particle number density. The equation $N^a = nu^a$ is valid on particle frame theory.

Hiscock and Lindblom [1] show that the stability of Israel-Stewart theory is equivalent to causality and hyperbolicity.

2.3. Equations of state

Equations of state of some thermodynamical variables can be derived from kinetic theory (Stewart [7], Caderni and Fabri [8]). To obtain detailed equations of state, we compare one component fluids in general relativity and in Newtonian theory as indicated by Ellis and Madsen [9]. There are some equations of state in the form of $\Sigma = \Sigma(p, v)$ where Σ is the specific internal energy density of the fluid, and Mason and Kgathi [10] have investigated the dependance of n and specific entropy S for a non-dissipative relativistic gas in collision-dominated equilibrium, where $S = S(\rho, n)$ and all the variables are defined by using the first law of thermodynamics.

$$\dot{S} = \frac{1}{nT}\dot{\rho} - \frac{\rho + p}{n^2 T}$$

In general any two of the thermodynamical scalars n, ρ, p, S and T are needed as independent variables. The relationship between the pressure p and energy density ρ is the linear barotropic equation of state:

$$p = (\gamma - 1)\rho$$

where

$$0 \leq \gamma \leq 2$$

The dependence of energy density was extended to the temperature T by Zakari and Jou [11] as given in equation (27) (see later). The Gibb's integrability condition

$$S, Tn = S, nT$$

becomes

$$n\frac{\delta T}{\delta n} + (p + \rho)\frac{\delta T}{\delta \rho} = T\frac{\delta p}{\delta \rho}$$

By taking $T = T(n, \rho)$ it follows on using the energy conservation equation and number conservation that

$$\dot{T} = -3H(n\frac{\delta T}{\delta n} + (p + \rho)\frac{\delta T}{\delta \rho})$$

This implies that :

$$\frac{\dot{T}}{T} = -3H(\frac{\delta p}{\delta \rho})_n$$

can give the equation of state for the pressure and temperature.

Chapter II

Numerical solution for cosmological model with a variable Λ

1. Introduction

There are many authors who have investigated the problem of a cosmological term in different ways such as Krauss *et al*, Croswell, Chen and Wu, Salim and Waga, Linde, and Carvalho *et al.* A wide range of observations suggest that the universe possesses a non-zero cosmological term (Krauss *et al* [13]). Other investigators have argued that because the cosmological term Λ is proportional to the energy density of vacuum, it has no need to be considered as a constant (Croswell [17]).

Some other authors consider various forms of the cosmological term. It is generally believed that Λ should decrease with the expansion of the universe as a function of the scale factor $R(t)$ as indicated by Chen and Wu [14]. The same applies to the Hubble factor (Carvalho *et al* [15] and Salim and Waga [16]), it decrease with the expansion of the universe as a function of scale factor $R(t)$.

In quantum field theory a cosmological term is regarded as a measure of the energy density of vaccum, and provides a force of repulsion opposing the gravitational pull between galaxies. Therefore the cosmological term corresponds to the energy density of vacuum. In the case of standard inflation, a universe with cosmological term would expand faster with time as shown by Croswell [17].

Further suggestions were made by Linde [18] that the cosmological term is a function of temperature related to spontaneous symmetry breaking processes. Calvao *et al* [19] construct a more realistic cosmological model with the dissipative bulk stress of cosmic fluid. Their work was extended by Mendez and Pavón [20] by numerical methods studying the evolution of the universe for $m = \frac{1}{2}$, $m = 1$ and $m = \frac{3}{2}$ using Eckart theory where $\Lambda = b/R^2(t)$

In this chapter we further investigate the numerical solution for the same values of m in Eckart theory for $\Lambda = 1/t^2$ and also in the truncated theory for $\Lambda = b/R^2(t)$. We also consider the graphical solution where $\gamma = 1, \gamma = 1.8$ and $\gamma = \frac{4}{3}$ for each value of m using Mathematica.

2. Field equations and model

We adopt a Friedmann-Lemaitre-Robertson-Walker (FLRW) universe, with energy momentum tensor filled with an imperfect fluid given by the equation :

$$T_{ab} = \rho u_a u_b + P_{eff} h_{ab} - \frac{\Lambda c^4}{8\pi G} g_{ab} \tag{1}$$

where ρ is the energy density, u_a is the four velocity, P_{eff} is the effective pressure, h_{ab} is the projection tensor defined by $h_{ab} = g_{ab} + u_a u_b$ and Λ is the cosmological term. The effective pressure splits into two parts:

$$P_{eff} = p + \Pi \tag{2}$$

where p is the hydrostatic equilibrium pressure and Π represent the bulk viscosity pressure [36].

The Einstein field equations for FLRW metrics reduces to:

$$3\frac{\dot{R}^2}{R^2} + 3\frac{k}{R^2} = \frac{8\pi G}{c^4}\rho + \Lambda \tag{3}$$

and

$$3\frac{\ddot{R}}{R} = -\frac{8\pi G}{c^4}(\frac{1}{2}\rho - \frac{\Lambda c^4}{8\pi G} + \frac{3}{2}(p+\Pi)) \tag{4}$$

The energy balance equation $u_a T^{ab}_{;b} = 0$ takes the form

$$\dot{\rho} = -3\frac{\dot{R}}{R}\rho - 3(p+\Pi)\frac{\dot{R}}{R} - \frac{\dot{\Lambda}c^4}{8\pi G} \tag{5}$$

The energy density of the matter field is not conserved because of the varying character of Λ, and the bulk viscosity pressure (Π) as indicated by Mendez and Pavón [20]. We assume the barotropic equation of the state as defined by Calvao *et al* [19], that is:

$$p = (\gamma - 1)\rho \tag{6}$$

where γ is a constant lying within the range $0 \leq \gamma \leq 2$.

3. Bulk viscous cosmologies in full causal theory

The number four-flux is given by

$$N^a = nu^a \tag{7}$$

when n is the number density. Then the energy entropy four-flux is given by

$$S^a = sN^a - \frac{\tau\Pi^2 u^a}{2\xi T} \tag{8}$$

where s is the specific entropy, $\tau \geq 0$ is the relaxation time for the bulk viscous stress, $\xi \geq 0$ is the coefficient of bulk viscosity and $T \geq 0$ is the temperature. Equation (7) obeys the law of conservation of four-flux

$$N^a_{;a} = 0 \tag{9}$$

The divergence of four-flux yields

$$S^{a}_{;a} = s_{;a}N^{a} + sN^{a}_{;a} - \frac{\tau\Pi\Pi_{;a}u^{a}}{\xi T} - \frac{\Pi^{2}}{2}(\frac{\tau u^{a}}{\xi T})_{;a} \tag{10}$$

Gibb's equation is given by

$$Tds = d(\frac{\rho}{n}) + pd(\frac{1}{n}) \tag{11}$$

From equation (11), by differentiating with respect to time, we have

$$nT\dot{s} = \dot{\rho} - \frac{\dot{n}}{n}\rho - \frac{\dot{n}}{n}p \tag{12}$$

By substituting equation (5) in equation (12), the resulting equation is

$$nT\dot{s} = -(\rho + p)(\Theta + \frac{\dot{n}}{n}) - \Pi\Theta \tag{13}$$

where

$$\Theta = 3H$$

By substituting equation (7) in equation (9), we get

$$\frac{\dot{n}}{n} = -\Theta \tag{14}$$

and by using equation (14) in equation (13), we obtain

$$nT\dot{s} = -\Pi\Theta \tag{15}$$

But from equations (7),(9) and (10), we obtain

$$Ts^{a}_{;a} = nT\dot{s} - \frac{\tau\Pi\dot{\Pi}}{\xi} - T\frac{\Pi^{2}}{2}(\frac{\tau u^{a}}{\xi T})_{;a}$$

which reduces by using equation (15) to

$$TS^{a}_{;a} = -\Pi\left(3H + \frac{\tau\dot{\Pi}}{\xi} + \frac{T\Pi}{2}(\frac{\tau u^{a}}{\xi T})_{;a}\right) \tag{16}$$

We require that the rate of generation of entropy should be non negative, i.e. $S^a_{;a} \geq 0$. Equation (16) then leads to the causal evolution equation for the bulk viscosity :

$$\Pi + \tau\dot{\Pi} = -3\xi H - \frac{\epsilon\tau\Pi}{2}(3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T}) \tag{17}$$

where H is the Hubble parameter as defined by

$$H = \frac{1}{3}\Theta = \frac{\dot{R}}{R} \tag{18}$$

and the dot represents a derivative with respect to time. We assume the *adhoc* equations

$$\xi = \alpha\rho^m \quad and \quad \tau = \frac{\xi}{\rho} \tag{19}$$

4. Solution at $\tau = 0$

For $\tau = 0$, equation (17) represents the non-causal or Eckart theory and takes the form

$$\Pi = -3\xi H \tag{20}$$

By using equations (3),(4),(6),(18),(19) and (20) we obtain:

$$\begin{aligned} 3\frac{\ddot{R}}{R} - \Lambda = {} & \frac{3}{2}(2 - 3\gamma)(\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} - \frac{\Lambda}{3}) + \\ & \frac{3^{m+2}}{2}\alpha\frac{8\pi G^{(1-m)}}{c^3}(\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} - \frac{\Lambda}{3})^m\frac{\dot{R}}{R} \end{aligned} \tag{21}$$

By considering the Law of Freese *et al* [21], we take the evolution of Λ as :

$$\Lambda = \frac{1}{t^2} \tag{22}$$

From equations (21) and (22) we obtain the evolution equation for R

$$3\frac{\ddot{R}}{R}-\frac{1}{t^2}=\frac{3}{2}(2-3\gamma)(\frac{\dot{R}^2}{R^2}+\frac{k}{R^2}-\frac{1}{3t^2})+$$
$$\frac{3^{m+2}}{2}\alpha\frac{8\pi G^{(1-m)}}{c^3}(\frac{\dot{R}^2}{R^2}+\frac{k}{R^2}-\frac{1}{3t^2})^m\frac{\dot{R}}{R} \tag{23}$$

The above equation (23) can lead to an equation for the deceleration parameter

$$q=-\frac{R\ddot{R}}{\dot{R}^2}$$

which is

$$q=-\frac{2-3\gamma}{2H^2}(H^2+\frac{k}{R^2}-\frac{1}{3t^2})-\frac{3^{m+1}}{H}\alpha(\frac{8\pi G}{c^3})^{1-m}$$
$$(H^2+\frac{k}{R^2}-\frac{1}{3t^2})^m-\frac{1}{3H^2t^2} \tag{24}$$

and the density parameter

$$\Omega=\frac{\rho}{3H^2}$$

as

$$\Omega=(1+\frac{k}{R^2H^2}-\frac{1}{3H^2t^2}) \tag{25}$$

Equations (3) and (19) can give the bulk viscosity coefficient expression as

$$\xi=(\frac{c^4}{8\pi G})^m(3\frac{\dot{R}^2}{R^2}+\frac{3k}{R^2}-\frac{1}{t^2})^m \tag{26}$$

Equations (24), (25) and (26) have the same form as the ones obtained by Mendez and Pavón [20], where they consider the law of Chen and Wu [14] for $\Lambda=b/R^2$.

5. Numerical solution for $\tau = 0$

We have numerically evaluated equations (23), (24), (25) and (26) for different values of m by means of Mathematica. We consider $\alpha(\frac{c^2}{8\pi G})^{m-1} = 1sec^{2m-1}$ as used by Mendez and Pavón [20]. We follow the same procedure used by Peebles [22], considering $t_0 = 17Gy, R_0 = 5.94 \times 10^{26}m$ and $H_0 = 100h$ where $h = 0.5$

5.1. For $m = \frac{1}{2}$

The equation (23) has a singularity at $t = 5.3112\times 10^{17}$, so we find no graphical solution. For $\Lambda = 1$ the present value of the deceleration parameter (q), the coefficient of bulk viscosity (ξ) and the density parameter were calculated for $k = -1, k = 0$ and $k = 1$ and the solutions are given in Table I below

Table I Present values for ξ, q and Ω for ($m = 1/2$)

k	$\xi_0(10^8 cm^{-1}s^{-1})$	Deceleration Parameter q_0	Density Parameter Ω_0
1	1.2197	-2.2174	0.6558
0	1.1258	-2.1040	0.5587
-1	1.0234	-1.9758	0.4617

From Table I above the values of the coefficient of the bulk viscosity happen to decrease as the value of k decreases, while the present value of q takes the negative form and corresponds to the same values obtain by Mendez and Pavón [20] for $\Lambda = b/R^2$ although they are not equal. This indicate the inflationary behaviour of the model. The change in all the values of ξ_0, q_0 and Ω_0 are in line with the ones obtained by Mendez and Pavón [20].

5.2. For $m = 1$

According to Barrow [23] this value correspond to a radiative fluid. For $\gamma = 1$ the present values of q as well as ξ and Ω are determined and shown below in Table II for each value of k.

Table II (Present values for ξ, q and Ω for $(m = 1)$).

k	$\xi_0(10^{-10}cm^{-1}s^{-1})$	Deceleration Parameter q_0	Density Parameter Ω_0
1	2.7734	-0.1134	0.6558
0	2.3630	-0.1619	0.5587
-1	1.9525	-0.2105	0.4617

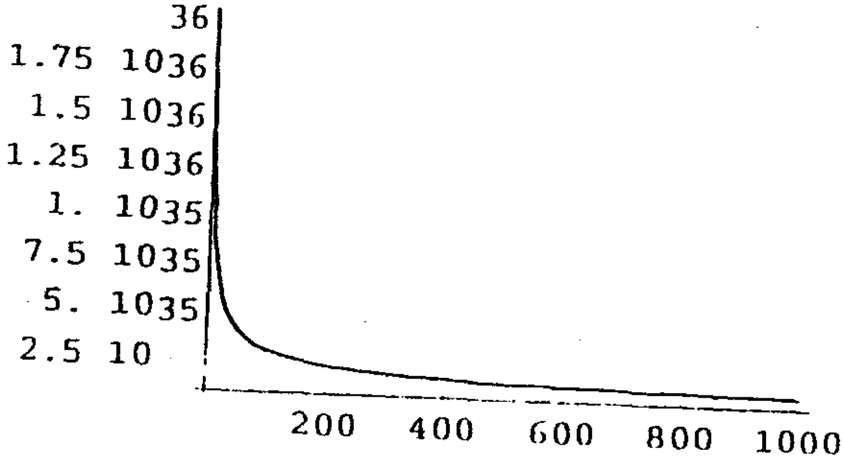
The values of ξ_0 and Ω_0 as given in Table II above are all decreasing as the value of k decreases, while the values of q_0 are negative and differ from the ones obtained by Mendez and Pavón [20]. In this case a graphical solution is possible and has a singularity at different points depending on the value of γ.

For $\gamma = \frac{4}{3}$ and $\gamma = 1$ at $k = -1$, $k = 0$ and $k = 1$, we have decreasing functions with time, as shown in (figure I) and (figure II) respectively, where the time t along the t axis is measured in Gy for all graphs. The graphs shift along the t-axis but maintain the same shape. For $\gamma = 1.8$ at $k = 1$ there is no graphical solution, at $k = 0$ we have a decreasing function with time (see figure III) and at $k = -1$ we have an increasing function with time (see figure IV).

5.3 For $m = \frac{3}{2}$

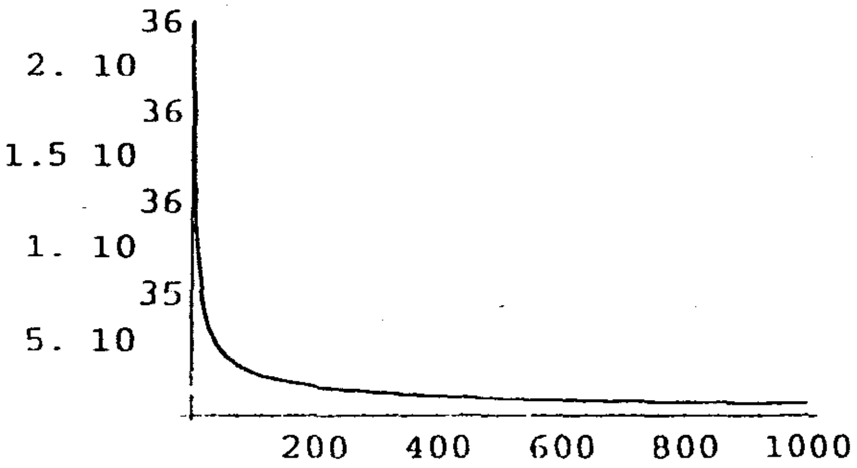
Again, at this point, we have no graphical solution because the boundary values for the interpolating function are too high (2051.49 and 1.5768×10^{18}).

Figure I : $R(t)$ versus t



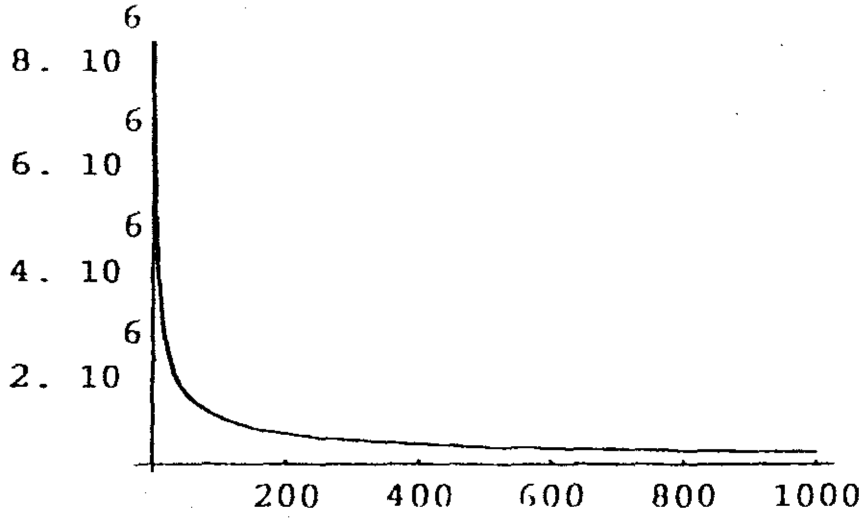
$\gamma = 4/3$ for all k for $m = 1$

Figure II : $R(t)$ versus t



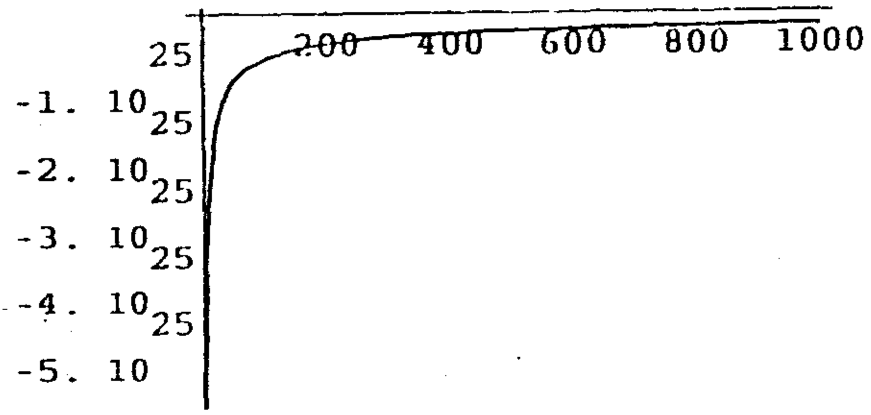
$\gamma = 1$ for all k for $m = 1$

Figure III : $R(t)$ versus t



$\gamma = 1.8$ for $k = 0$ for $m = 1$

Figure IV : $R(t)$ versus t



$\gamma = 1.8$ for $k = -1$ for $m = 1$

The present values of ξ, q and Ω are determined and given in Table III below.

Table III (Present values for ξ, q and Ω for $(m = 3/2)$.

k	$\xi_0(10^{-28}cm^{-1}s^{-1})$	Deceleration Parameter q_0	Density Parameter Ω_0
1	6.3064	-0.1134	0.6558
0	4.9595	-0.1619	0.5587
-1	3.7251	-0.2105	0.4617

5.4. For $m = 2$

The present values of ξ_0, q_0 and Ω_0 are calculated and given in Table IV.

Table IV (Present values for ξ, q and Ω for $(m = 2)$.

k	$\xi_0(10^{-45}cm^{-1}s^{-1})$	Deceleration Parameter q_0	Density Parameter Ω_0
1	1.4340	-0.1134	0.6558
0	1.0409	-0.1619	0.5587
-1	0.7107	-0.2105	0.4617

From Table IV it is clear that the values of ξ_0 also decrease as the values of k change from positive to negative values, and also decreases as the value of m increases (see Table I, II and III). The values of q_0 and Ω_0 are the same as the ones obtained in Table II and Table III for $m = 1$ and $m = \frac{3}{2}$. A graphical solution is possible for $\gamma = \frac{4}{3}$ and $\gamma = 1$. It has the same shape as the ones obtained in figure I and figure II for all values of k. No graph is obtained for $\gamma = 1.8$.

5.5 Conclusions

In our solutions the present values of the deceleration parameter (q) are negative for all values of m and k. This confirms the inflationary nature of the solutions, where the universe is still under accelerated expansion. These values are *slightly less than the values obtained* by Mendez and Pavón [20], *except for* $m = 1$ and $m = \frac{3}{2}$ where they obtain positive values. The values of ξ_0 decrease as the values of m increase.

For $m = 1$ and $k = 0$ the function $R(t)$ decrease as time increases, which is different from the ones obtained by Mendez and Pavón [20] where they have increasing functions with time. The shapes of all the graphs for $m = 1$, $k = -1$, $k = 0$ and $k = 1$ are the same for $\gamma = 1, \gamma = \frac{4}{3}$ and $\gamma = 1.8$. For $m = 1, k = -1$ and $\gamma = 1.8$ the function $R(t)$ is an increasing function with initial singularity and it is below the Einstein Static universe, therefore at this point we have an initial singularity and the universe is expanding forever with time.

For $\gamma = 1$ and $\gamma = \frac{4}{3}$ where $m = 1$ we have the same graph for all the values of k and they each represent a decreasing function with different initial singularity. For $\gamma = 1.8$ the only possible graphical solution is when $k = 0$ which is also a decreasing function with time, while when $k = 1$ we find that there is no graphical solution because the values at the initial singularity are too high. For $m = 2$ the solution is similar to the one obtained for $m = 1$ for both $\gamma = 1$ and $\gamma = \frac{4}{3}$, but for $\gamma = 1.8$ there is no graphical solution obtained. Therefore the behaviour of the Friedmann-Lemaitre-Robertson-Walker models depend upon the form of the cosmological term.

6. Evolution equation for $\epsilon = 0$

Mendez and Pavón [20], only work out the numerical solution of the same model for $\Lambda = b/R^2$ where $\tau = 0$ as in the case above in section 4. Their solution ignores the effect of the relaxation time. In this section we are considering the same definition of the cosmological term and we take $\epsilon = 0$ which represent the truncated theory, although this theory also has some drawback of neglecting the effect of the temperature but it partly involves the effect of the relaxation time.

We consider the cosmological term as given by :

$$\Lambda = \frac{b}{R^2} \tag{27}$$

where b is a fixed pure number of order unity as suggested by Chen and Wu [14]. For $\epsilon = 0$ equation (17) reduces to

$$\Pi + \tau\dot{\Pi} = -3\xi H \tag{28}$$

By using equations (3), (4), (6), (19) and (28), we obtain the evolution equation for R

$$\begin{aligned} &\frac{\dddot{R}}{R} + 3(\gamma - 1)\frac{\dddot{R}\dot{R}}{R^2} + (2 - 3\gamma)(\frac{\dot{R}^3}{R^3} + \frac{k - \frac{b}{3}}{R^3}\dot{R}) + \frac{2b\dot{R}}{3R^3} \\ &\qquad -\frac{1}{\alpha}\left(\frac{3C^4}{8\pi G}\right)^{1-m}\left(\frac{\dot{R}^2}{R^2} + \frac{k - \frac{b}{3}}{R^2}\right)^{1-m} \\ &(\frac{2 - 3\gamma}{2}(\frac{\dot{R}^2}{R^2} + \frac{k - \frac{b}{3}}{R^2}) + \frac{b}{3R^2} - \frac{\ddot{R}}{R}) = \frac{9}{2}(\frac{\dot{R}^2}{R^2} + \frac{k - \frac{b}{3}}{R^2})\frac{\dot{R}}{R} \end{aligned} \tag{29}$$

From equation (29) we can have the deceleration parameter as given by :

$$q = \frac{1}{3(\gamma-1)}\frac{\ddot{R}R^2}{\dot{R}^2} + \frac{2-3\gamma}{3(\gamma-1)}(1+\frac{k-\frac{b}{3}}{\dot{R}^2}) + \frac{2b}{9(\gamma-1)\dot{R}^2} -$$

$$\frac{3}{2(\gamma-1)}(1+\frac{k-\frac{b}{3}}{\dot{R}^2}) - \frac{1}{3^m\alpha(\gamma-1)}\left(\frac{c^4}{8\pi G}\right)^{1-m}$$

$$\left(\frac{\dot{R}^2}{R^2}+\frac{k-\frac{b}{3}}{R^2}\right)^{1-m}(\frac{2-3\gamma}{2}(\frac{R}{\dot{R}}+\frac{k-\frac{b}{3}R}{\dot{R}^3})+\frac{bR}{3\dot{R}^3}-\frac{\ddot{R}R^2}{\dot{R}^3}) \tag{30}$$

Equations (29) and (30) are more complex if we compare with the same equation for τ as given by equations (23) and (24). It is different from the one obtained by Mendez and Pavón [20]

7. Numerical solution for $\epsilon = 0$

For our numerical solution we take

$q_0 = 0.5$

$H_0 = 100 hkm.s^{-1}Mpc^{-1}$

$t_0 = 17Gy$

and $H_0 = 5.94 \times 10^{28}$ where $h = 0.5$ and we also take

$$\alpha\left(\frac{c^2}{8\pi G}\right)^{m-1} = 1sec^{2m-1}$$

The above equation (29) is also evaluated by using Mathematica for $b = 1$ and it works for $\gamma = \frac{4}{3}$ and $\gamma = 1.8$ at different values of m. The solutions are represented graphically.

7.1. For $m = \frac{1}{2}$

For this value of m, we have obtained graphical solutions for $\gamma = 1.8$ where $k = -1, k = 0$ and $k = 1$. The graphs are similar to the one obtained by Mendez and Pavón [20] for $\tau = 0$ and the function is an increasing function with time. From the graph the age of the universe is $55Gy$ and is slightly larger than the one obtained by Mendez and Pavón [20] (see figure V).

7.2. For $m = 1$

In this case, we were able to find graphical solutions for $\gamma = 1.8$ and $\gamma = \frac{4}{3}$ for all values of k. For $\gamma = 1.8$, the age of the universe is $51Gy$ at all values of k, which corresponds to the values obtained by Mendez and Pavón for the same γ where $m = \frac{1}{2}$ (see figure VII), while for $\gamma = \frac{4}{3}$, the graphs have the same shape as the ones obtained when $\gamma = 1.8$ but shift to negative values along the R(t) axis (see figure VI and figure VIII) for all values of k.

The shift is caused by the small value of the deceleration parameter q as compared to the one obtained by Mendez and Pavón [20]. The decrease in the value of q results from the additional term $1/t^2$ as indicated in equation (15).

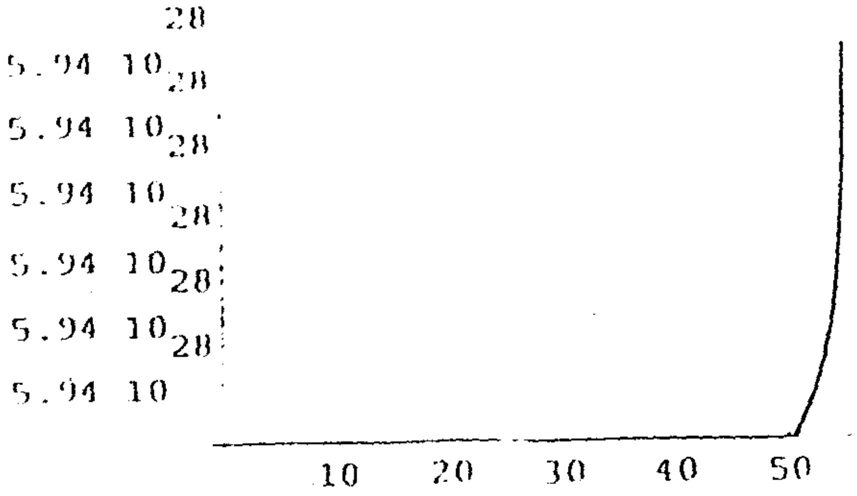
7.3 For $m = \frac{3}{2}$

This solution corresponds to a fluid of fundamental strings. Graphical solutions were obtained for $\gamma = \frac{4}{3}$ and $\gamma = 1.8$ for all values of k. These graphs are similar to those obtained when $m = 1, k = 0$ and $k = 1$ (see figure IX and figure X)

7.4 Conclusion

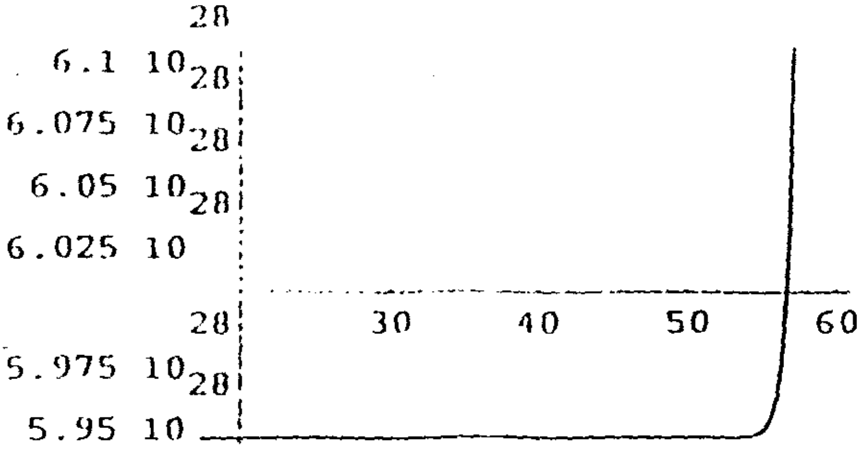
All the graphs obtained have the same shape as the ones obtained by Mendez and Pavón [20] for $m = \frac{1}{2}$. This indicates that all these solutions do not have much impact to our present universe, since the graph shifts to negative values along the R(t) axis because of the small value of the deceleration parameter.

Figure V : $R(t)$ versus t



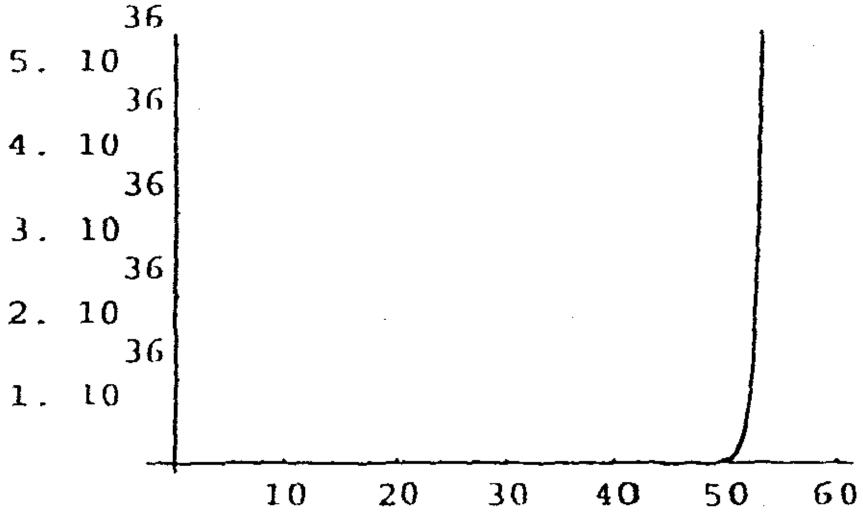
γ 1.8 for k 1 for m 1/2

Figure VI : $R(t)$ versus t



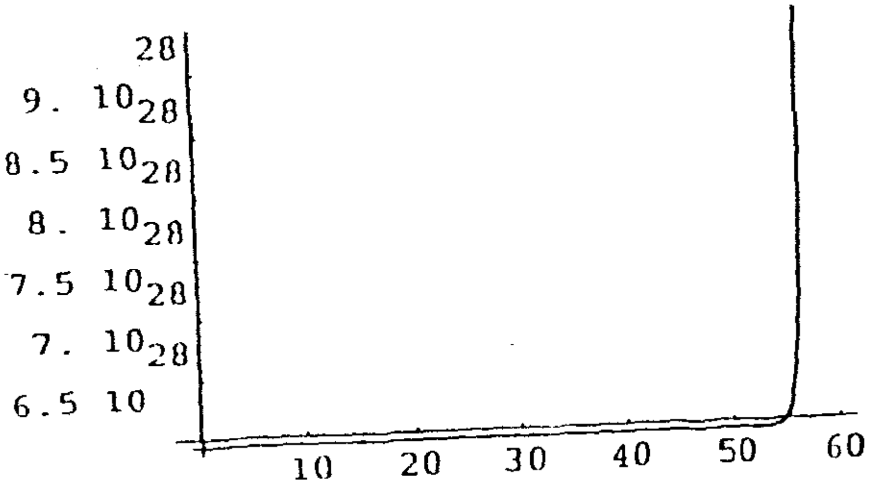
γ 4/3 for k 0, 1 1 and for m 1

Figure VII : $R(t)$ versus t



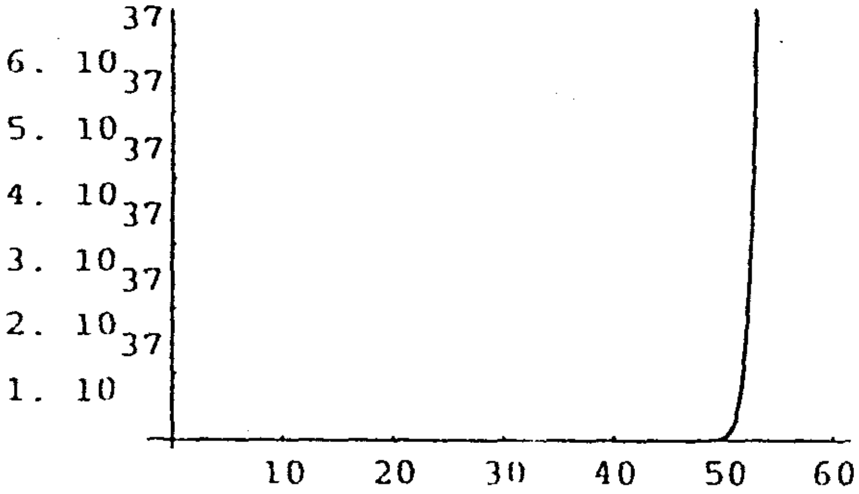
$\gamma = 1.8$ for $k = 0, +1$ for $m = 1$

Figure VIII : $R(t)$ versus t



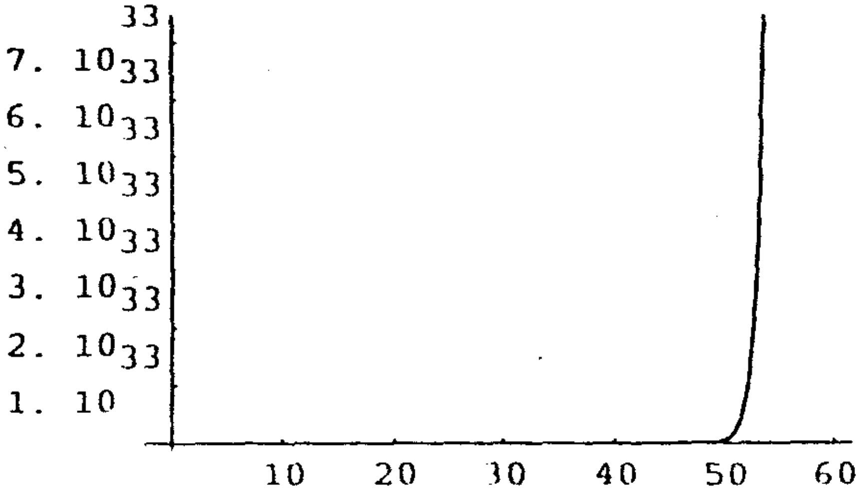
$\gamma = 4/3$ for $k = -1$ for $m = 1$

Figure IX : $R(t)$ versus t



$\gamma = 1.8$ for all $k = -1$ for $m = 1$

Figure X : $R(t)$ versus t



$\gamma = 4/3$ for all k for $m = 3/2$

Chapter III

Exact solution of cosmological model with a variable Λ

1. Introduction

The role played by viscosity and the consequent dissipative mechanism in cosmology has been discussed by many authors. The expansion rate of the universe is slowed down by the scalar field from exponential to polynomial so that there is enough time for the universe to complete the phase transition from the inflationary to the radiation dominated phase. Dissipative effects such as viscosity are of enormous importance in the early stages of the evolution of the universe, particularly before the time of nucleosynthesis (Grøn [24]).

A number of observations, e.g., type IA supernovae, now compellingly suggest that the universe possesses a non zero cosmological term (Krauss and Turner [13]). The cosmological term represents energy which is combined with the matter of the universe. For standard inflation, a universe with a cosmological term would expand faster with time because of the push from the cosmological term (Croswell [17]). The cosmological term could be a function of time in the spatially homogenous expanding universe.

Any model of the universe should yield a lifetime greater than that of the oldest objects in it, so it is difficult for the Friedmann-Lemaitre Robertson-Walker (FLRW) models without the cosmological term to have an age of the universe greater than of the oldest stars (Bagla *et al* [25] ; Fukuyama *et al* [26] and Davies [27]).

The first solution of cosmological models with time dependent G and Λ were obtained by Bertolami [29, 30] which were extended by Abdussattar and Vishwakama [31]. Also Arbab [32] found several solutions similar to ones obtained by Berman [33] and Kalligas *et al* [34], claiming that energy is conserved. The solution was modified by Singh *et al* [35], where they obtain the energy density ρ as a decreasing function of time, with energy conservation.

Most of these investigations are based on Eckart theory. Beesham, Ghosh and Nkosi studied some solutions with variable Λ in Eckart theory and in the truncated theory, which have been submitted for publication. This is attached as an appendix.

In this chapter we examine the full causal theory which solves the previously mentioned problems. An exact solution of an FLRW model is obtained and discussed, where $k = 0$ and the cosmological term is a function of the Hubble parameter given by $\Lambda = 3\beta H^2$. Also the expressions for the parameters Λ, ρ, τ, ξ and T as functions of time are obtained and discussed.

2. Field equations

The energy momentum tensor for a fluid with bulk viscosity in equation (1) takes the form

$$T_{ab} = \rho u_a u_b + p_{eff} h_{ab} - \Lambda g_{ab} \tag{31}$$

where ρ is the energy density, u_a is the four velocity, p_{eff} is the effective pressure, Λ is the cosmological term, and h_{ab} is the projection tensor defined by

$$h_{ab} = g_{ab} + u_a u_b$$

and we are using units such that $8\pi \mathrm{G} = c = 1$.

In a spatially flat FLRW cosmological model ($k = 0$) the metric is defined as:

$$ds^2 = -dt^2 + R^2(t)(dx^2 + dy^2 + dz^2) \tag{32}$$

From equations (31) and (32), the Einstein equations

$$G_{ab} = T_{ab}$$

become

$$\rho = 3H^2 - \Lambda \tag{33}$$

and

$$\dot{H} = -H^2 - \frac{1}{6}\rho - \frac{1}{2}(p + \Pi) + \frac{\Lambda}{3} \tag{34}$$

where

$$H = \frac{\dot{R}}{R}$$

The conservation equation of energy momentum yields

$$\dot{\rho} = -3(\rho + p\Pi)H$$

but is not independent. We assume a linear barotropic equation of state (6), where γ is a constant within the range $0 \leq \gamma \leq 2$.

By using equations (6), (33) and (34) we obtain the equation for the bulk viscous stress

$$\Pi = -2\dot{H} - 3\gamma H^2 + \gamma\Lambda \tag{35}$$

From equations (17) and (35), we obtain the evolution equation for H

$$\begin{aligned} 2\dot{H} + 2H^2 &= -2\tau\ddot{H} - (6\gamma\tau + 3\epsilon\tau)H\dot{H} - \\ &\epsilon\tau\dot{w}\dot{H} - \frac{9}{2}\epsilon\tau\gamma H^3 - (3\gamma - 2 + \frac{3}{2}\epsilon\tau\dot{w}\gamma)H^2 \\ &+(3\xi + \frac{3}{2}\epsilon\tau\gamma\Lambda)H + \tau\gamma\dot{\Lambda} + (\gamma + \frac{1}{2}\epsilon\tau\gamma\dot{w})\Lambda \end{aligned} \tag{36}$$

where

$$\dot{w} = \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T}$$

The temperature evolution equation as used by Zakari and Jou [11]; Maartens [40] and Zimdahl [41] is given by :

$$T = T_0\rho^a \tag{37}$$

For radiative gas a different thermodynamic approach exists depending on whether the particle number density is taken as an independent variable or not. In the case of power-law for the temperature there exists a relationship between γ and a as :

$$a = \frac{(\gamma - 1)}{\gamma} \tag{38}$$

and T_0 is a constant. Following Arbab [32], we take the value of the cosmological term to be

$$\Lambda = 3\beta H^2 \tag{39}$$

where β is a constant. With the help of equations (19), (33), (36) and (39) we can construct an evolution equation for H as:

$$\begin{aligned} 2\ddot{H}H + (6\gamma + 3\epsilon - 3(a+1)(1-\beta)\epsilon\gamma)\dot{H}H^2 + \\ 2(3(1-\beta))^{1-m}\alpha^{-1}\dot{H}H^{3-2m} - 2(a+1)\epsilon\dot{H}^2 + \\ (\frac{9}{2}\gamma\epsilon(1-\beta) - 9(1-\beta))H^4 + \\ 3^{2-m}\alpha^{-1}\gamma(1-\beta)^{2-m}H^{5-2m} = 0 \end{aligned} \tag{40}$$

As discussed by Maartens [40], the equation in the form of (40) is consistent with exponential inflation with $\Lambda = 0$ and $H = H_0 = constant$. Equation (40) gives an inflationary solution if $\beta = 1$ and if $m \neq 0$, we obtain $\rho = 0$, $\xi = 0$ and $\tau = 0$ which corresponds to a perfect fluid solution. For $H = H_0 = constant$ and $\beta \neq 1$ we can also have an inflationary solution ($\rho = \Lambda = \xi = \tau = \Pi = constant$), i.e., $R \propto exp(H_0 t)$. For realistic models for which ρ is a decreasing function and ξ is not vanishing we require $0 < \beta < 1$. The coefficient of the bulk viscosity also does not vanish during the radiation epoch if $\beta < 0$. Therefore the bulk viscosity remains constant during the inflationary phase, which corresponds to the solution obtained by Wolf [42].

3. Exact solution

Equation (40) admits a power law solution for the Hubble factor H in the form

$$H(t) = H_0 t^x \tag{41}$$

From equation (41) we obtain

$$\dot{H} = H_0 x t^{x-1} \quad \mathit{and} \quad \ddot{H} = H_0 x(x-1)t^{x-2} \tag{42}$$

where H_0 is a constant. By using equations (40), (41) and (42) we obtain

$$\begin{aligned} 2x^2(x-1)t^{2x-2} + Axt^{3x-1} + Bxt^{4x-2mx-1} - \\ Cx^2t^{2x-2} + Dt^{4x} + Et^{5x-2xm} = 0 \end{aligned} \tag{43}$$

where

$$A = 6\gamma - 3\epsilon - 6\beta\gamma - 3(a+1)(1-\beta)\epsilon\gamma$$

$$B = 2(3(1-\beta))^{1-m}\alpha^{-1}$$

$$C = 2(a+1)\epsilon$$

$$D = \frac{9}{2}\gamma\epsilon(1-\beta) - 9(1-\beta)$$

$$E = 3^{2-m}\alpha^{-1}\gamma(1-\beta)^{2-m}$$

3.1. For $m = \frac{1}{2}$ and $\epsilon = 1$ (full causal theory)

Equations (38) and (43) yields the following equations

$$2x^2(x-1) - 2(\frac{2\gamma - 1}{\gamma})x^2 = 0 \tag{44}$$

$$6\gamma - 3 - 6\beta\gamma - 3(2\gamma - 1)(1 - \beta) + 2\,(3(1 - \beta))^{\frac{1}{2}}\,\alpha^{-1} = 0 \tag{45}$$

$$\frac{9}{2}\gamma(1 - \beta) - 9(1 - \beta) + \sqrt{27}\alpha^{-1}\gamma(1 - \beta)^{\frac{3}{2}} = 0 \tag{46}$$

Equation (44) is satisfied for

$$x = 1 \tag{47}$$

and

$$\gamma = \frac{1}{2} \tag{48}$$

By using equations (45), (46), (47) and (48), we obtain that

$$\beta = 3 \quad \textit{and} \quad \alpha = -\frac{(2\sqrt{-2})}{(3\sqrt{3})} \tag{49}$$

From equation (41) and (47), we have

$$H = H_0 t \tag{50}$$

and then from equations (39), (49) and (50), we obtain

$$\Lambda = 9H_0^2 t^2 \tag{51}$$

which reduces equation (33) to

$$\rho = -6H_0^2 t^2 \tag{52}$$

With the help of equations (47), (48), (49) and (52), equations (6), (19), (35) and (37) become

$$p = 3H_0^2t^2 \tag{53}$$

$$\Pi = 3H_0^2t^2 - 2H_0 \tag{54}$$

$$\xi = -\frac{4}{3}H_0t \tag{55}$$

$$\tau = \frac{2}{9H_0}t^{-1} \tag{56}$$

$$T = -\frac{T_0}{6H_0^2}t^{-2} \tag{57}$$

The energy density ρ is a decreasing function with time since it contains a negative sign. It corresponds to the value obtained by Banerjee and Beesham [43] where $\Lambda = 0$. The temperature behaviour depends upon the value of T_0. For $T_0 < 0$ the temperature is a decreasing function with time which corresponds to the value obtained by Zakari and Jou [11], Maartens [40] and Banerjee and Beesham [43] in the FLRW model for $\Lambda = 0$. The value of α is a complex number, so the solution above does not have much physical impact.

3.2. For $m = \frac{3}{2}$ and $\epsilon = 1$

The equation (43) yields the following equations

$$2x^2(x-1) - 2(\frac{2\gamma - 1}{\gamma})x^2 = 0$$

$$6\gamma - 3 - 6\beta\gamma - 3(2\gamma - 1)(1-\beta) = 0$$

$$2\left(3(1-\beta)\right)^{\frac{-1}{2}}\alpha^{-1} = 0$$

$$\frac{9}{2}\gamma(1-\beta) - 9(1-\beta) = 0$$

$$\sqrt{3}\alpha^{-1}\gamma(1-\beta)^{\frac{1}{2}} = 0$$

which can be solved and a solution is

$$x = \frac{5}{2}, \beta = 0, \gamma = 2$$

and

$$\frac{1}{\alpha} \simeq 0$$

where

$$\alpha \gg 1$$

By using the solution in equations (6), (33), (35), (37), (39) and (41) we obtain

$$H = H_0 t^{\frac{5}{2}}$$

$$\Lambda = 0$$

$$\rho = 3H_0 t^{\frac{5}{2}}$$

$$p = 3H_0 t^{\frac{5}{2}}$$

$$\Pi = -6H_0^2 t^5 - 5H_0 t^{\frac{3}{2}}$$

$$T = \sqrt{3H_0 T_0} t^{\frac{5}{4}}$$

The energy density ρ is an increasing function with time while the temperature is a decreasing function with time depending on the value of T_0. The bulk viscosity pressure takes a negative value and it is a decreasing function with time.

Equation (50) yields the scale factor R

$$R = exp\left(\frac{H_0t^2 + 2C_0}{2}\right) \tag{58}$$

where C_0 is a constant of integration. Therefore equation (40) can admit an exponential solution for the scale factor R for both $m = \frac{1}{2}$ and $m = \frac{3}{2}$.

4. Conclusion

In this chapter we have investigated the Einstein equations for the viscous flat FLRW universe in the full causal theory, where the gravitation parameter G is constant and the cosmological term Λ is a variable. We have obtained exact solutions and expressions for $\Lambda, \rho, p, \Pi, \tau, \xi$ and T as functions of time. For $m = \frac{1}{2}$ we can conclude that ρ is an increasing function with time, but is having a negative value, while the temperature is a decreasing function with time for $T_0 < 0$. This indicates that the model is not physical. The cosmological term varies as the inverse square of time, which matches its natural units. This supports the views of various authors, viz., Bartolami [30], Berman [44] and Berman *et al* [45]. The thermodynamic pressure (p), the bulk viscous stress (Π) and the coefficient of bulk viscosity (ξ) are increasing functions with time.

For $m = 3/2$, ρ is an increasing function with time and the model is physical since ρ is positive. The cosmological term is vanishing and the bulk viscosity stress is negative.

Chapter IV

Qualitative analysis of cosmological models with a variable Λ

1. Introduction

It is known that dissipative processes are vitally important in the evolution of relativistic fluids both in cosmology and high energy astrophysical phenomena. Some investigations were conducted in Eckart theory, which suffers from serious pathologies and drawbacks. Most papers dealing with viscous and heat conducting cosmological models use the truncated theory without stating clearly what the implications of such simplification may be (Mendez and Trigner [46]).

In this chapter we analyze non-flat FLRW models with viscosity and variable Λ in the full causal theory by setting up a dynamic system following the same method used by Coley *et al* [48]. The equilibrium points are obtained and interpreted for both $\epsilon = 0$ and $\epsilon = 1$. An exact solution is obtained for $\epsilon = 1$.

2. Dynamic system

For the application of qualitative analysis of the equations, the dynamical system is required. The relaxation time (τ) and the bulk viscosity coefficient (ξ) are important in order to have a closed dynamical system where the temperature law for T plays a major role. According to Coley *et al* [48], the relaxation time and bulk viscosity coefficient are considered as proportional to powers of ρ and this also applies to the temperature (T). The dimensionless density parameter is given by

$$z = \frac{\rho}{3H^2} \tag{59}$$

and assuming that ξ/H and $1/(\tau H)$ are proportional to powers of z, we have

$$\frac{\xi}{H} = 3xi_0 z^n \tag{60}$$

$$\frac{\tau^{-1}}{H} = bz^p \tag{61}$$

Maartens and Mendez [49] and also Zimdahl [50] have pointed out that the increase of the temperature and energy density during the inflationary period in the full causal theory seems to be an unavoidable feature of any bulk viscosity driven inflation and should follow from the qualitative analysis of the general equations of the full theory. Let the temperature be

$$T = T_0 z^p \rho^a \tag{62}$$

For $p = 0$, equation (61) reduces to

$$\frac{1}{\tau} = bH \tag{63}$$

and equation (62) takes the form of equation (37). For $a = 1/4$ the local equilibrium state of the expanding viscous fluid is in thermal radiation. For $z = 1$ the bulk viscosity coefficient and the relaxation rate are determined by the expansion rate ($\xi \propto H$). According to Coley *et al* [48], the transformation of the system to a new time coordinate is possible by defining the dimensionless viscous pressure as y and the new time variable as $\bar{t}$ where

$$y = \frac{\Pi}{H^2} \tag{64}$$

and

$$\frac{d\bar{t}}{dt} = H \tag{65}$$

By assuming that $8\pi G = c = 1$, from equations (3) and (39) we have

$$(1-\beta)H^2 = \frac{1}{3}\rho - \frac{k}{R^2} \tag{66}$$

and from equations (4), (6) and (39) we deduce that

$$\dot{H} = \left(\frac{\beta}{2} - 1\right)H^2 - \frac{1}{6}(3\gamma - 2)\rho - \frac{1}{2}\Pi \tag{67}$$

By using equations (6) and (26) in equation (5) we get

$$\dot{\rho} = -3\gamma\rho H - 6\beta H\dot{H} - 3\Pi H \tag{68}$$

Equation (59) can be differentiated with respect to time and using equations (65), (67) and (68) we obtain

$$z' = ((\beta - 1) + z)\left(y + (3\gamma - 2)z - \beta - \frac{\beta}{z + (\beta - 1)}\right) \tag{69}$$

where the prime represent the derivative with respect to the new time $\bar{t}$ and equation (64) yields

$$\dot{y} = -2\frac{\dot{H}}{H}y \tag{70}$$

By using equations (17), (59), (60), (63) and (64) in equation (70) one obtains the following equation

$$\dot{y} = -9z_0 z^n H - 2y\left(\frac{b}{2} + (\frac{\beta}{2} - 1) - \frac{1}{2}(3\gamma - 2)z + \frac{1}{2}y\right)H - \frac{3}{2}\epsilon y H - \epsilon y \dot{w} \tag{71}$$

where

$$z_0 = \xi_0 b \quad \mathit{and} \quad \dot{w} = \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T} \tag{72}$$

From equations (37), (60), (63), (65), (67), (72) and (77) one gets

$$\begin{gathered} y' = -9z_0 z^n - \frac{1}{2} y \,(2b + 3\epsilon) - y\,((1+a)\epsilon + 2) \\ \left((\frac{\beta}{2} - 1) - \frac{1}{2}(3\gamma - 2)z - \frac{1}{2}y - \frac{1}{2}(a+n)\epsilon \frac{'xy}{x} \right) \end{gathered} \tag{73}$$

Equation (69) and (73) form a plane autonomous system of ordinary differential equations (ODE) for y and z.

3. Equilibrium point of the system at $\epsilon = 0$

The value of $\epsilon = 0$ corresponds to the truncated theory where the effective source of pathological behaviour in T is terminated. Equation (73) reduces to:

$$y' = -9z_0 z^n - by - 2y \left((\frac{\beta}{2} - 1) - \frac{1}{2}(3\gamma - 2)z - \frac{1}{2}y \right) \tag{74}$$

For $z = 1 - \beta$ the above equation yields

$$y' = y^2 - (b - 3\gamma(1-\beta) - \beta)\, y - 9z_0(1-\beta)^n \tag{75}$$

Equilibrium exists when $y' = 0$. We then have two values of y from equation (65)

$$y^+ = \frac{1}{2}\,(b - 3\gamma(1-\beta) - \beta) + S_1$$

and

$$y^- = \frac{1}{2}\,(b - 3\gamma(1-\beta) - \beta) - S_1$$

where

$$S_1 = \sqrt{b - 3\gamma(1-\beta) - \beta + 36z_0(1-\beta)^n}$$

Therefore we have the first set of equilibrium points given by

$$\Big((1-\beta), y^+\Big) \quad and \quad \Big((1-\beta), y^-\Big) \tag{76}$$

The point$((1-\beta), y^+)$ represents a nodal point and it is stable, while the point $((1-\beta), y^-)$ is a saddle point and it is asymptotically stable. For $S_1 = 0$ the only equilibrium point is at a point where

$$y = b - 3\gamma(1-\beta) - \beta$$

This point depends on the value of b, the constant for the relaxation rate. According to Maartens [40], the point is negative since for the viscous expansion to be non-thermalising, we must have $\tau^{-1} < H$, and the constraint on the relaxation time parameter is $b < 1$. There are other equilibrium points of equation (69) at $z \neq (1-\beta)$ which depends on values of n. From equation (69) we have the new values of y denoted by $\bar{y}$ given by the equation :

$$\bar{y} = \beta - (3\gamma - 2)\bar{z} - \frac{\beta}{\bar{z} + (\beta - 1)} \tag{77}$$

By using equation (77) in equation (75) one obtains

$$\begin{aligned} &9z_0\bar{z}^{n+2} + 18z_0(1-\beta)\bar{z}^{n-1} + 9z_0(1-\beta)^2\bar{z}^n \\ &\qquad -d_0\bar{z}^3 - d_1\bar{z}^2 - d_2\bar{z} - d_3 = 0 \end{aligned} \tag{78}$$

where

$$d_0 = (3\gamma - 2)$$

$$\left[b + 2(\frac{\beta}{2} - 1) - 2(3\gamma - 2)(\beta - 1) + \beta - 2(2 - \beta - 3\gamma\beta)\right]$$

$$d_1 = 2(\beta-1)(3\gamma-2)b - b\beta - 2\beta(\frac{\beta}{2}-1) +$$

$$4(3\gamma-2)(\frac{\beta}{2}-1) - \beta(3\gamma-2) - (3\gamma-2)^2(\beta-1)^2 +$$

$$2\beta(3\gamma-2)(\beta-1) + (3\beta - 3\gamma\beta +$$

$$3\gamma-2)^2 - 2\beta(\beta-2)(3\gamma-2)$$

$$d_2 = (3\gamma-2)(\beta-1)^2 b - b\beta - 4\beta(\frac{\beta}{2}-1)(\beta-1)$$

$$+2(3\gamma-2)(\frac{\beta}{2}-1)(\beta-1)^2 + 2\beta(\frac{\beta}{2}-1) -$$

$$\beta(\beta-1)(3\gamma-2) + \beta(3\gamma-2)$$

$$(\beta-1)^2 + 2\beta(\beta-2)(3\beta-3\gamma\beta+3\gamma-2)$$

$$d_3 = b\beta(\beta-1) - b\beta(\beta-1)^2 - 2\beta(\frac{\beta}{2}-1)$$

$$(\beta-1)^2 - 2\beta(\frac{\beta}{2}-1) + (\beta(\beta-2))^2$$

3.1. For $n = 0$

For $n = 0$ equation (78) reduces to

$$\bar{z}^3 - \frac{9z_0 - d_1}{d_0}\bar{z}^2 - \frac{18z_0(1-\beta) - d_2}{d_0}\bar{z} -$$

$$\frac{9z_0(1-\beta)^2 - d_3}{d_0} = 0 \qquad (79)$$

Then from equation (79) there are three sets of equilibrium points :

$$(\bar{z}_1, \bar{y}_1), (\bar{z}_2, \bar{y}_2) \quad and \quad (\bar{z}_3, \bar{y}_3) \qquad (80)$$

which correspond to

$$\bar{z}_1 = A + B + \frac{9z_0 - d_1}{3d_0}$$

$$\bar{y}_1 = \beta - (3\gamma - 2)\bar{z}_1 - \frac{\beta}{z_1 + (\beta - 1)}$$

$$\bar{z}_2 = wA + w^2B + \frac{9z_0 - d_1}{3d_0}$$

$$\bar{y}_2 = \beta - (3\gamma - 2)\bar{z}_2 - \frac{\beta}{z_2 + (\beta - 1)}$$

$$\bar{z}_3 = w^2A + wB + \frac{9z_0 - d_1}{3d_0}$$

$$\bar{y}_3 = \beta - (3\gamma - 2)\bar{z}_3 - \frac{\beta}{z_3 + (\beta - 1)}$$

The values of A and B are given by

$$A = \left(\frac{J + \frac{1}{27d_0^3}C}{54d_0^3}\right)^{\frac{1}{3}}$$

and

$$B = \left(\frac{J - \frac{1}{27d_0^3}C}{54d_0^3}\right)^{\frac{1}{3}}$$

where

$$J = 9d_0(d_1 - 9d_0)(d_2 - 18z_0(1 - \beta)) -$$

$$2(d_1 - 9z_0)^3 - 27d_0^2(d_3 - 9z_0(1 - \beta)^2)$$

$$C = \sqrt{(3d_0d_2 - 54d_0z_0(1 - \beta)^2)^3 + F}$$

and

$$F = \frac{1}{4}(F1 - F2 + F3)^2$$

where

$$F1 = 27d_0^2 d_3 (1-\beta)^2$$

$$F2 = (9d_0 d_2 - 162 z_0 d_0 (1-\beta))(d_1 - 9z_0)$$

$$F3 = 2(d_1 - 9z_0)^2$$

The parameter w is given by

$$w = -\frac{1}{2} + \frac{i\sqrt{3}}{2} \quad and \quad w^3 = 1$$

The set of equilibrium points depends on the values of $\bar{z}_1, \bar{z}_2$ and $\bar{z}_3$.

3.2. For n =1

By taking $n = 1$ equation (78) reduces to

$$\bar{z}^3 + b_1 \bar{z}^2 + c\bar{z} + d = 0 \tag{81}$$

where

$$b_1 = \frac{18z_0(1-\beta) - d_1}{9z_0 - d_0}$$

$$c = \frac{9z_0(1-\beta)^2 - d_2}{9z_0 - d_0}$$

$$d = \frac{-d_3}{9z_0 - d_0}$$

Equation (81) is a cubic polynomial and it has three values of $\bar{z}$. Therefore the three equilibrium points are :

$$(\bar{z}_1, \bar{y}_1), (\bar{z}_2, \bar{y}_2) \ and \ (\bar{z}_3, \bar{y}_3) \tag{82}$$

where

$$\bar{z}_1 = A + B - \frac{b_1}{3}$$

$$\bar{y}_1 = \beta - (3\gamma - 2)\bar{z}_1 - \frac{\beta}{\bar{z}_1 + (\beta - 1)}$$

$$\bar{z}_2 = wA + w^2B - \frac{b_1}{3}$$

$$\bar{y}_2 = \beta - (3\gamma - 2)\bar{z}_2 - \frac{\beta}{\bar{z}_2 + (\beta - 1)}$$

$$\bar{z}_3 = w^2A + wB - \frac{b_1}{3}$$

$$\bar{y}_3 = \beta - (3\gamma - 2)\bar{z}_3 - \frac{\beta}{\bar{z}_3 + (\beta - 1)}$$

and the values of A and B are

$$A = (A1 + A2)^{1/3}$$

where

$$A1 = \frac{9b_1c - 2b_1^3 - 27d}{54}$$

and

$$A2 = \frac{1}{27}\sqrt{(3c - b_1^2)^3 + \frac{1}{4}(27d - 9b_1c + 2b_1^3}$$

and

$$B = (B1 - B2)^{1/3}$$

where

$$B1 = \frac{9b_1c - 2b_1^3 - 27d}{54}$$

and

$$B2 = \frac{1}{27}\sqrt{(3c - b_1^2)^3 + \frac{1}{4}(27d - 9b_1c + 2b_1^3}$$

Also these equilibrium points are saddle points and have the same behaviour as the equilibrium point in equation (80)

3.3. For $n = 2$

Equation (78) reduces to a quartic equation for $n = 2$. Therefore it has four sets of equilibrium points. The quartic equation take the form

$$\bar{z}^4 + \bar{b}_1\bar{z}^3 + \bar{c}\bar{z}^2 + \bar{d}\bar{z} + e = 0 \tag{83}$$

where

$$\bar{b}_1 = \frac{18z_0(\beta - 1) - d_0}{9z_0}$$

$$\bar{c} = \frac{9z_0(\beta - 1)^2 - d_1}{9z_0}$$

$$\bar{d} = -\frac{d_2}{9z_0}$$

$$e = -\frac{d_3}{9z_0}$$

Then the four sets of equilibrium points are

$$(\bar{z}_1, \bar{y}_1), (\bar{z}_2, \bar{y}_2), (\bar{z}_3, \bar{y}_3) \ and \ (\bar{z}_4, \bar{y}_4) \tag{84}$$

where

$$\bar{z}_1 = \frac{1}{2}\left(M_1 + M_2 + M_3\right)$$

$$\bar{z}_2 = \frac{1}{2}\left(M_1 - M_2 - M_3\right)$$

$$\bar{z}_3 = \frac{1}{2}\left(-M_1 + M_2 - M_3\right)$$

$$\bar{z}_4 = \frac{1}{2}\left(-M_1 - M_2 + M_3\right)$$

$$M_1 = \sqrt{A + B - \frac{1}{12}(8\bar{c} - 3\bar{b}_1^2)}$$

$$M_2 = \sqrt{wA + w^2B - \frac{1}{12}(8\bar{c} - 3\bar{b}_1^2)}$$

$$M_3 = \sqrt{w^2A + wB - \frac{1}{12}(8\bar{c} - 3\bar{b}_1^2)}$$

and

$$\bar{y}_1 = \beta - (3\gamma - 2)\bar{z}_1 - \frac{\beta}{z_1 + (\beta - 1)}$$

$$\bar{y}_2 = \beta - (3\gamma - 2)\bar{z}_2 - \frac{\beta}{z_2 + (\beta - 1)}$$

$$\bar{y}_3 = \beta - (3\gamma - 2)\bar{z}_3 - \frac{\beta}{z_3 + (\beta - 1)}$$

$$\bar{y}_4 = \beta - (3\gamma - 2)\bar{z}_4 - \frac{\beta}{z_4 + (\beta - 1)}$$

The values of A and B are

$$A = \left(\frac{108(8\bar{d} - 4\bar{b}_1\bar{c} + \bar{b}_1^3)^2 + M_4 + 13824\sqrt{S_2}}{13824}\right)$$

and

$$B = \left(\frac{108(8\bar{d} - 4\bar{b}_1\bar{c} + \bar{b}_1^3)^2 + M_4 - 13824\sqrt{S_2}}{13824}\right)$$

where

$$S_2 = \frac{1}{2916}\left(\frac{-108(8\bar{d} - 4\bar{b}_1\bar{c} + \bar{b}_1^3)^2 + M_4}{256}\right)$$

$$+\frac{1}{729}\left(\frac{-3(256e - 64\bar{b}_1\bar{d} + 16\bar{b}_1^2\bar{c} - 3\bar{b}_1^4) - (8\bar{c} - 3\bar{b}_1^2)^2}{64}\right)^3$$

$$M_4 = (8\bar{c} - 3\bar{b}_1^2)^3 - 9(8\bar{c} - 3\bar{b}_1^2)(256e - 64\bar{b}_1\bar{d} + 16\bar{b}_1^2\bar{c} - 3\bar{b}_1^4)$$

The equilibrium point $(\bar{z}_1, \bar{y}_1)$ is a nodal point and is stable, while the other three point $(\bar{z}_2, \bar{y}_2)$, $(\bar{z}_3, \bar{y}_3)$ and $(\bar{z}_4, \bar{y}_4)$ are saddle points and are asymptotically stable for all values of

$$\sqrt{A + B - \frac{1}{12}(\bar{c} - 3\bar{b}_1)} > 0$$

$$\sqrt{wA + w^2B - \frac{1}{12}(\bar{c} - 3\bar{b}_1)} > 0$$

and

$$\sqrt{w^2A + wB - \frac{1}{122}(\bar{c} - 3\bar{b}_1)} > 0$$

while $(\bar{z}_1, \bar{y}_1)$ is a saddle point and asymptotically stable for all the values of

$$\sqrt{A + B - \frac{2}{1}12(\bar{c} - 3\bar{b}_1)} < 0$$

$$\sqrt{wA + w^2B - \frac{1}{12}(\bar{c} - 3\bar{b}_1)} < 0$$

and

$$\sqrt{w^2A + wB - \frac{1}{12}(\bar{c} - 3\bar{b}_1)} < 0$$

while the other points are nodal points and stable.

4. Equilibrium point of the system at $\epsilon = 1$

For this value of ϵ, we have the full causal theory. Equation (73) reduces to

$$y' = -9z_0 z^n - \frac{1}{2}y(2b+3) -$$

$$y(a+3)\left((\frac{\beta}{2}-1) - \frac{1}{2}(3\gamma-2)z - \frac{1}{2}y\right) \tag{85}$$

For $z = 1-\beta$, equation (85) yields

$$y' = \frac{1}{2}(a+3)y^2 -$$

$$\left(\frac{1}{2}(2b+3) + (a+3)(\frac{3}{2}\beta\gamma - \frac{3}{2}\gamma - \frac{1}{2}\beta\right)y -$$

$$9z_0(1-\beta)^n \tag{86}$$

For $y' = 0$, the equation (86) reduces to a quadratic equation and the two values of y at the equilibrium points are

$$y^+ = \frac{\left(\frac{1}{2}(2b+3) + (a+3)(\frac{3}{2}\beta\gamma - \frac{3}{2}\gamma - \frac{1}{2}\beta)\right) + \sqrt{T_1}}{a+3}$$

and

$$y^- = \frac{\left(\frac{1}{2}(2b+3) + (a+3)(\frac{3}{2}\beta\gamma - \frac{3}{2}\gamma - \frac{1}{2}\beta)\right) - \sqrt{T_1}}{a+3}$$

where

$$T_1 = \frac{1}{2}(2b+3) + (a+3)(\frac{3}{2}\beta\gamma - \frac{3}{2}\gamma - \frac{1}{2}\beta) + 18z_0(1-\beta)^n(a+3)$$

Therefore the set of equilibrium points are

$$\left((1-\beta), y^+\right) \ \textit{and} \ \left((1-\beta), y^-\right) \tag{87}$$

The point $((1-\beta), y^{+})$ is a nodal point and is stable while $((1-\beta), y^{-})$ is a saddle point and asymptotically stable. If $T_1 = 0$, the only equilibrium point exists when

$$y = \frac{\frac{1}{2}(2b+3) + (a+3)(\frac{3}{2}\beta\gamma - \frac{3}{2}\gamma - \frac{1}{2}\beta)}{a+3}$$

The equilibrium point depends upon the value of the constant a. For $a > 1$, the equilibrium point is positive which is totally different from the one obtained by Coley *et al* [48] in a flat FLRW model. For $z \neq (1-\beta)$ the new value of y at equilibrium is

$$\bar{y} = \beta - (3\gamma - 2)\bar{z} - \frac{\beta}{\bar{z} + (\beta - 1)} \tag{88}$$

where $\bar{z}$ represent the new value of z at equilibrium. The polynomial in equation (78) reduces to

$$y' = -9z_0 - y(c_0 - c_1 z - c_2 y + c_3 y z^{-1} - c_4 z^{-1}) \tag{89}$$

where

$$c_0 = \frac{1}{2}(2b+3) + (a+3)(\frac{\beta}{2} - 1) + \frac{1}{2}$$

$$(n+a)(\beta-1)(3\gamma-2) - \frac{1}{2}(a+n)\beta$$

$$c_1 = \frac{(a+3)(3\gamma-2) - (a+n)(3\gamma-2)}{2}$$

$$c_2 = \frac{3-n}{2}$$

$$c_3 = \frac{1}{2}(n+a)(\beta-1)$$

$$c_4 = \frac{1}{2}\beta(a+n)(\beta-2)$$

By using equation (88) in equation (89) one obtain a new polynomial in terms of $\bar{z}$

$$9z_0\bar{z}^{n+3} + 18z_0(\beta-1)\bar{z}^{n+2} + 9z_0(\beta-1)\bar{z}^{n+1} + d_0\bar{z}^5 - d_1\bar{z}^4 + d_2\bar{z}^3 + d_3\bar{z}^2 + d_4\bar{z} + (\beta^2-2\beta)c_3 = 0 \tag{90}$$

By taking $n = 0$ and $n = 1$, we obtain the same solution as in equation (78) for $n = 1$ and $n = 2$, respectively. So the behaviour of the equilibrium points are the same.

5. Exact dynamic solution

From equation (64) the viscous pressure is given by

$$\Pi = yH^2 \tag{91}$$

By using $y = y^+$ as defined in equations (78) and (87) equation (91) yields

$$\Pi = y^+H^2 \tag{92}$$

For $z = (1-\beta)$ equation (59) reduces to

$$\rho = 3(1-\beta)H^2 \tag{93}$$

which yields

$$\dot{\rho} = 6(1-\beta)H\dot{H} \tag{94}$$

By using equations (68), (91) and (94) we get

$$\frac{\dot{H}}{H^2} = -\frac{y^+ + 3\gamma(\beta-1)}{2}$$

which can be integrated to give

$$H(t) = \frac{2}{(y^{+} + 3\gamma(1-\beta))\,t} \tag{95}$$

By considering the Hubble parameter $H = \dot{R}/R$, equation (95) gives

$$R(t) = R_0 t^{A} \tag{96}$$

where

$$A = \frac{2}{(y^{+} + 3\gamma(1-\beta))}$$

With the help of equation (95), equations (39), (66) and (92) respectively yield

$$\Lambda(t) = \frac{12\beta t^{-2}}{(y^{+} + 3\gamma(1-\beta))^{2}} \tag{97}$$

$$\rho(t) = \frac{12(1-\beta)t^{-2}}{(y^{+} + 3\gamma(1-\beta))^{2}} + \frac{3k}{R_0^2 t^{2A}} \tag{98}$$

$$\Pi(t) = \frac{4y^{+}t^{-2}}{(y^{+} + 3\gamma(1-\beta))^{2}} \tag{99}$$

At the point $y = y^{-}$ the above solution defined by equations (95), (96), (97), (98) and (99) respectively take the form

$$H(t) = \frac{2(t-t_0)^{-1}}{(y^{-} + 3\gamma(1-\beta))} \tag{100}$$

$$R(t) = (t-t_0)^{\frac{2}{y^{-}+3\gamma(1-\beta)}} \tag{101}$$

$$\Lambda(t) = \frac{12\beta(t-t_0)^{-2}}{(y^{-} + 3\gamma(1-\beta))^{2}} \tag{102}$$

$$\rho(t) = \frac{12(1-\beta)(t-t_0)^{-2}}{(y^{-} + 3\gamma(1-\beta))^{2}} + \frac{3k}{R_0^2 t^{\frac{4}{y^{-}+3\gamma(1-\beta)}}} \tag{103}$$

and

$$\Pi = \frac{4y^{-}(t-t_0)^{-2}}{(y^{-} + 3\gamma(1-\beta))^{2}} \tag{104}$$

3. Conclusion

In this chapter we considered the equilibrium points for both TIS and FIS of the FLRW model with variable Λ and the exact solution. At $z = 1 - \beta$ there exist two equilibrium points, $(1-\beta, y^{+})$ which is a nodal point and is stable and $(1-\beta, y^{-})$ which is a saddle point and asymptotically stable for both theories. In the case of FIS the value of y^{+} and y^{-} depend upon the value of a. In both cases the equilibrium point (0;0) depends upon the value of n, it also represents the Milne model in the case where $\Lambda = 0$.

For $z \neq 1 - \beta$, the equilibrium point depends upon the value of n for both theories. For $n = 0$ and $n = 1$ there are three equilibrium points and all are saddle points depending upon the values of A and B as indicated below for equation (80) and (82). For TIS as shown in the case of $n = 2$ there are four equilibrium points three are saddle points and asymptotically stable while the other one is a nodal point and is stable for all values of

$$\sqrt{A + B - \frac{1}{12}(\bar{c} - 3\bar{b}_1)} > 0$$

$$\sqrt{wA + w^2 B - \frac{1}{12}(\bar{c} - 3\bar{b}_1)} > 0$$

and

$$\sqrt{w^2 A + wB - \frac{1}{122}(\bar{c} - 3\bar{b}_1)} > 0$$

For $\Lambda = 0$ in the exact solution, the value of ρ is having an additional term $3k/R_0^2 t^{2A}$ which makes it different from the one obtained by Coley *et al* [48] where $\Lambda = 0$. For $k = 0$, ρ can be expressed in terms of Λ and the equation will be

$$\rho = \frac{12t^{-2}}{(y^{+} + 3\gamma(1-\beta))^{2}} - \Lambda$$

at a point where y $= y_{+}$. For $\Lambda = 0$ the equation above will give similar results as the one obtained by Coley *et al* [48].

Chapter V

Conclusion

In this work we have investigated some cosmological solutions with bulk viscosity and variable Λ. We first investigate the numerical solution using Eckart theory where $\tau = 0$. To study the numerical solution the present values of time, scale factor R and the hubble parameter H were considered as initial conditions at different values of m, where the present values of the coefficient of bulk viscosity (ξ_0), deceleration parameter q_0 and density parameter Ω_0 were calculated. The density parameter is the same for all values of m when k change from positive to negative values, while the coefficient of bulk viscosity decreases as the value of m increases, and also decreases when k change from positive to negative values (see table I, II, II and IV).

The value of the deceleration parameter increases as k changes for $m = 1/2$ (see table I), and decreases for $m = 1$, $m = 3/2$ and $m = 2$ as shown in tables II, III and IV. All the values of q_0 are negative, which indicate that the model is still under accelerated expansion. Graphs were obtained only for $m = 1$ and $m = 2$ and have the same shape (see figure I and figure II).

Secondly we find the numerical solution by using the truncated theory where graphs were obtained for $m = 1/2$, $m = 1$ and $m = 3/2$. All the graphs have the same shape if $\gamma = 1.8$ (see figure V, figure VII and figure IX). For $\gamma = 4/3$ the graphs are not the same, they shift below to negative values of $R(t)$ as the values of m increase, but maintain the same shape.

We find numerical solutions in both the Eckart theory and the truncated theory for different values of m. Future work could be do the numerical study of cosmology with variable Λ in the full causal theory.

Also we have considered the exact solution of the model at $m = 1/2$ and $m = 3/2$ using the full causal theory. The expressions for $H, \Lambda, \rho, p, \Pi, \tau, \xi$ and T as functions of time were obtained. For $m = 1/2$ the values of H, Λ, p and Π are increasing functions with time, while the values of ρ, τ, ξ and T are decreasing, also the temperature (T) and the energy density (ρ) are negative. This indicates that the model is not physical at $m = 1/2$. The values of H, ρ, p and T are positive increasing functions for $m = 3/2$, while the value of Π is negative and decreasing with time. The cosmological term becomes zero.

We find some exact cosmological solutions in the full causal theory with variable Λ under some reasonable assumptions. Because of the highly non-linear nature of the field equations, exact solutions are difficult to find. Future work could be to determine further exact solutions.

Finally we consider the qualitative solution of the model by using the truncated theory and the full causal theory. Also the equilibrium points were obtained and discussed at $z = 1 - \beta$ and $z \neq 1 - \beta$. The equilibrium point (0,0) exists in both theories and it depends upon the value of the constant n. An exact dynamic solution was also obtained, where the values of R, H, Λ, ρ and Π are positive decreasing functions with time and the expression for ρ will give similar values as the ones obtained by Coley *et al* [48] for $\Lambda = 0$.

Although we have determined the equilibrium points and their nature, future work could be to plot the trajectories in phase plane diagrams as has been done by Coley *et al* [48] with $\Lambda = 0$.

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APPENDIX

PREPRINT

Viscous cosmological models with a variable cosmological term

By

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Viscous Cosmological Models With a Variable Cosmological Term

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Abstract

Einsteins field equations for a Friedmann-Robertson-Walker universe filled with a dissipative fluid with a variable cosmological term Λ described by Eckart theory and extended irreversible thermodynamics are considered. Exact solutions of the field equations for the flat case have been obtained. In Eckart's theory, the cosmological term varies in proportion to the inverse square of time. There is a period of hyperinflation or superinflation when the bulk viscosity coeffecient is constant whereas the energy density and cosmological term grow enormously. Other solutions corresponding to a particular choice of parameters are presented as well.

Key words: Cosmology, bulk viscosity, variable cosmological term.
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1 Introduction

In recent years, there has been a tendency to regard the comological constant as one of the most fundamental physical entities [1]. In fact, one of the most important problems in cosmology is the cosmological constant problem [2, 3], which attempts to explain the small value of the effective constant at present. In the context of quantum field theory a cosmological term corresponds to the energy density of vacuum. The possibility of a non zero Λ term, in particular, has surfaced lately in connection with the age problem of the universe [4, 5]. A wide range of observations suggests that

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the universe possesses a non zero cosmological constant [4]. Some authors [6, 7] have argued that because the cosmological term is proportional to the energy density of vacuum, it has no need to be a constant but, on the contrary, it should decrease with the expansion of the universe as a function of the scale factor [8, 9, 10] and of the Hubble factor as well [11, 12]. The latest measurements of the Hubble parameter [4] point to an intrinsic fragility of standard (photon conserving) Friedmann-Robertson-Walker (FRW) cosmology in such way that models without a cosmological constant seem to be effectively ruled out. For further details of models with variable Λ but with no bulk viscosity, we refer to Overduin and Cooperstock [15].

To construct a more realistic cosmological model, one can add to the picture the bulk viscous effects of the cosmic fluid. There are many circumstances during the evolution of the universe, especially at early stages, in which bulk viscosity could arise [16, 17, 18] and could lead to an effective mechanism of entropy production. Bulk viscosity can explain the anomalously high entropy per baryon ratio in the contemporary universe [19, 20]. It is well known that viscous effects could drive an inflationary period independently of the details of phase transition of grand unified theories [20]. The general criterion of bulk viscosity was given by Weinberg [3]. The role of dissipative effects on the evolution of universe has been investigated by many authors [21]-[26] and are largely based on the well known first order relativistic theory of non-equilibrium thermodynamics due to Eckart [27]. Eckart theory is widely used, but however, it has two important shortcomings, namely (i) it predicts an infinite speed for the propagation of viscous pulses and, (ii) it presents some pathalogical instabilities [28]. In order to overcome these difficulties, one can consider a higher order theory, i.e., extended irreversible thermodynamics (henceforth, EIT) [30, 29]. This is more general than conventional Eckart theory and has a number of advantages which are discussed by Hiscock and Lindblom [31]. For a review, see the article by Jou *et al* [32].

Recently, Chimento and Jakubi [33, 34] have found exact solutions to Einstein's gravitational field equations in a homogeneous universe filled with a causal viscous fluid source for bulk viscosity index $m = \frac{1}{2}$ and without a cosmological term ($\Lambda = 0$). Very recently, Mak and Harko [35] have obtained a general exact solution in a similar formalism. In these investigations, the evolution of the universe is reduced mathematically to a linear ordinary differential equation leading consequently to a general solution in parametric or non parametric form.

In this paper we consider bulk viscous FRW models with a variable cosmological term proportional to the square of the Hubble parameter, $\Lambda \sim H^2$, and the bulk viscosity proportional to a power law of the energy density $\xi \sim \rho^m$. We have obtained exact solutions both in Eckart theory and in EIT. The behaviour of the cosmological term, energy density and bulk viscosity is discussed. Other exact solutions to the field equations corresponding to particular values of the parameter are presented too. We find that some of the results discussed by Mak and Harko [35] and Arbab and Beesham [36] in the full causal theory (details of full causal theory is discussed in [37, 38]) arise in our case as well.

The organisation of the paper is as follows: In Section 2 we introduce our main assumptions. Section 3 is devoted to the derivation of the differential equations for the evolution of the Hubble factor (both in Eckart theory and EIT) and in finding the exact solutions of the models. The paper ends with our conclusions in section 4.

2 Basic Equations

Let us consider a homogeneous and isotropic universe represented by the FRW metric

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi\right] \tag{1}$$

where R(t) is the scale factor and $k = 0, \pm 1$ is the curvature constant corresponding to flat, closed and open spatial sections respectively.

Einstein's field equations with time dependent cosmological term are

$$R_{ab} - \frac{1}{2} R\, g_{ab} = 8\pi G\, T_{ab} + \Lambda(t)\, g_{ab} \tag{2}$$

where R_{ab} is the Ricci tensor, g_{ab} the metric tensor, R the Ricci scalar, G the gravitational constant and T_{ab} the energy momentum tensor.

For a perfect fluid T_{ab} is given by

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} \tag{3}$$

Here ρ is the energy density, p the pressure and u_a the 4-velocity of the fluid. To include the effects of viscosity we have to replace the perfect pressure p by an effective pressure p_{eff}, which is given by

$$p_{eff} = p + \Pi$$

Here p is the perfect fluid contribution and Π is the bulk viscous stress.

The field equations (2) , together with the energy momentum tensor defined by equation (3) and with the FRW metric (1) for $k = 0$ now become

$$3H^2 = \rho + \Lambda \tag{4}$$

$$3(H^2 + \dot{H}) = -\frac{1}{2}(\rho + 3p + 3\Pi) + \Lambda \tag{5}$$

where $H = \dot{R}/R$ is the Hubble factor, and an overdot denotes differentiation with respect to time t. We used units such that $c = 8\pi G = 1$.

To complement equations (4) and (5) one needs an equation of state for p

$$p = (\gamma - 1)\rho \tag{6}$$

where γ is a constant, $\gamma = 1$ corresponds to dust and $\gamma = 4/3$ corresponds to radiation, and a constitutive equation for Π. When using the EIT formalism the equation reads [30]

$$\Pi + \tau\dot{\Pi} = -3\xi H \tag{7}$$

The quantity ξ is the bulk viscous coefficient which cannot be negative otherwise the principle of entropy increase would be violated. The quantity $\tau \geq 0$ is the relaxation coefficient for the transient bulk viscous effects. For $\tau = 0$ one gets back the non-causal Eckart theory. To proceed further, one must specify the dependence of ξ and τ on the energy density. It is standard to assume [37, 38] the following *ad hoc* laws:

$$\xi = \alpha\rho^m \qquad \textit{and} \qquad \tau = \frac{\xi}{\rho} \tag{8}$$

where m is a constant. If $m = 1$, equation (8) may correspond to a radiative fluid, whereas $m = 3/2$ may correspond to a string-dominated universe [20]. However, more realistic models [39] are based upon m lying in the regime $0 \leq m \leq 1/2$.

Follwing Arbab [23, 24, 25] and Singh *et al* [22], we take the variation of Λ to be of the form

$$\Lambda = 3\beta H^2 \tag{9}$$

where β is a constant.

3 Cosmological Solutions

3.1 Eckart theory ($\tau = 0$)

Let us recall briefly previous results for $\tau = 0$ which corresponds to Eckart theory. Beesham [21] studied a universe consisting of a cosmological constant ($\Lambda \sim t^{-2}$) and bulk viscosity. He showed Berman's model [42] could be viscous for $n = 1/2$. Arbab [25] has discussed a viscous model with variable G and Λ. Several solution were obtained and have been shown equivalent to those of Berman and Kalligas *et al* [40, 41]. In the same view Singh *et al* [22] obtained a similar solution with a different conservation law from the one used by Arbab [25].

Using eqs. (4)-(9) we get, for the evolution of the Hubble factor H, the equation

$$\dot{H} + \frac{3}{2}(1-\beta)\gamma H^2 = \frac{9\delta}{2}(1-\beta)H^{2m+1} \tag{10}$$

where $\delta = \alpha[3(1-\beta)]^{m-1}$ is a constant.

- $m \neq 1/2$

Eq. (10) admits a de Sitter solution

$$H = \left[\frac{\gamma}{3\delta}\right]^{\frac{1}{2m-1}} = H_0 \tag{11}$$

It follows from eqs (4)-(9), that $\Lambda = const.$, $\rho = const.$, $\xi = const.$, $\Pi = const.$ For $\beta = 0$ this solution reduces to a solution which has been discussed by several authors [26, 33, 34, 37, 42] (with no cosmological term ($\Lambda = 0$)). We see that bulk viscosity, the energy density and the cosmological term remain constant during this inflationary period.

- $m = 1/2$

We consider the solution of eq. (10) corresponding to a bulk viscosity coefficient proportional to the square root of the density or proportional to the Hubble factor ($\xi \sim H$).

Eq. (10) can be integrated to yield

$$H = \frac{1}{at + C} \tag{12}$$

so that the scale factor becomes

$$R = R_0(at + C)^{1/a} \tag{13}$$

Where R_0 and C denote constants of integration. Using eqs. (4), (5), (8) and (9) we get

$$\rho = \frac{3(1-\beta)}{(at+C)^2} \tag{14}$$

$$\Lambda = \frac{3\beta}{(at+C)^2} \tag{15}$$

$$\Pi = \frac{-2a - 3(1-\beta)\gamma}{(at+C)^2} \tag{16}$$

$$\xi = \frac{\alpha\sqrt{3(1-\beta)}}{at+C} \tag{17}$$

where we have denoted constant $a = 3/2(1-\beta)(\gamma - 3\delta)$. The condition $0 < \beta < 1$ ensures that the solutions are real and physical. This solution, for $C = 0$, is similar to the one discussed by Mak and Harko [35] in the full causal theory. The universe described by eqs. (12)-(16), for $C = 0$, starts from a singular state with infinite energy density. During its evolution the energy density, bulk viscosity and cosmological term decrease. The variation $\Lambda \sim t^{-2}$, obtained here is consistant with Kalligas *et al*, Beesham [21] and Arbab [23] and thought to be fundamental.

- $m = 0$ and $\gamma = 0$

Eq (10), integrates to

$$H = const.exp\left[\frac{3\delta}{2}t\right] \tag{18}$$

so that

$$R = const.exp\left[const.exp\left[\frac{3\delta}{2}t\right]\right] \tag{19}$$

which is hyperinflationary [26]. During this epoch the cosmological term and energy density grows very enormously. This solution was obtained by Arbab [25] in a Bianchi I model with variable G and Λ, but in his case the density remains constant during this period.

- $\beta = 1$

From eqs (9) and (10), we get $H = H_0 = \sqrt{\Lambda/3}$ so that $R = const.exp((\sqrt{\Lambda/3}t)$, i.e., an inflationary period. During this period we see that $\Lambda = const.$, $\rho = 0$. It follows that $\xi = 0$ and also $\Pi = 0$, and we recover a solution in [36].

3.2 EIT ($\tau \neq 0$)

In this case the evolution equation, for the Hubble factor, takes the form

$$\ddot{H} + 3(1-\beta)\gamma H\dot{H} + \frac{1}{\delta}H^{2-2m}\dot{H} + \frac{3}{2}(1-\beta)H^3\left[\frac{\gamma}{\delta}H^{1-2m} - 3\right] = 0 \tag{20}$$

- $m \neq 1/2$

We note that in this case also eq. (20) admits a deSitter $H = H_0$ solution with $\ddot{H} = \dot{H} = 0$ and H_0 given by eq. (11). This is an inflationary expansion of viscous origin [43].

- $m = 1/2$ and $B \neq 0$

In this case eq. (20) takes the form

$$\ddot{H} + AH\dot{H} - BH^3 = 0 \tag{21}$$

where $A = 3\gamma(1-\beta) + 1/\delta$, $B = 3/2(3-\gamma/\delta)(1-\beta)$ are constants. By means of a transformation [33, 35]

$$H^2 = y \qquad \eta = A\int Hdt \tag{22}$$

eq. (21) turns into a linear differential equation

$$\frac{d^2y}{d\eta^2} + \frac{dy}{d\eta} - by = 0 \tag{23}$$

where we have denoted $b = 2B/A^2$. We obtain the general solution of (23) in the parametric form

$$H(\eta) = \left[C_1 exp(\mu^+\eta); + C_2 exp(\mu^-\eta)\right]^{\frac{1}{2}} \tag{24}$$

$$\Lambda(\eta) = 3\beta\left[C_1 exp(\mu^+\eta); + C_2 exp(\mu^-\eta)\right] \tag{25}$$

$$t - t_0 = \int\left[C_1 exp(\mu^+\eta); + C_2 exp(\mu^-\eta)\right]^{\frac{-1}{2}} d\eta \tag{26}$$

where t_0, C_1 and C_2 are arbitrary constants of integration, and $\mu^{\pm} = (-1\pm\sqrt{1+4b})/2$ are the roots of the characteristic equation of the differential equation (23). For the solution to be real the quantity in the square root must be positive. When either of the integration constants, C_1 or C_2, vanishes the solution can be expressed as a one-parameter family of solutions as in [33].

- $m = 1/2$ and $B = 0$

If we choose the parameter $B = 0$, i.e., if γ and δ fulfil the restriction that $\gamma = 3\delta$ then eq. (10) admits a de Sitter solution $H = H_0$, where H_0 is an arbitary constant. For $H = H_0 = 0$ ($R = const.$) we get a Minkowskian solution. On the other hand, for $H_0 \neq 0$, eq. (21) can be integrated to yield

$$H = \frac{C_1}{\sqrt{A}} tanh \left[\frac{\sqrt{A} C_1 (t - 2C_2)}{2} \right] \tag{27}$$

where C_1 and C_2 are integration constants. In this case we have

$$R = R_0 cosh^{\frac{2}{A}} \left[\frac{\sqrt{A} C_1 (t - 2C_2)}{2} \right] \tag{28}$$

$$\rho = 3(1 - \beta) \frac{C_1^2}{A} tanh^2 \left[\frac{\sqrt{A} C_1 (t - 2C_2)}{2} \right] \tag{29}$$

$$\Lambda = 3\beta \frac{C_1^2}{A} tanh^2 \left[\frac{\sqrt{A} C_1 (t - 2C_2)}{2} \right] \tag{30}$$

$$\begin{aligned} \Pi = & \frac{-C_1^2}{A} tanh^2 \left[\frac{\sqrt{A} C_1 (t - 2C_2)}{2} \right] - \\ & 3(1 - \beta) \frac{C_1^2}{A} tanh^2 \left[\frac{\sqrt{A} C_1 (t - 2C_2)}{2} \right] \end{aligned} \tag{31}$$

$$\xi = \alpha \sqrt{3(1 - \beta)} \frac{C_1}{\sqrt{A}} tanh \left[\frac{\sqrt{A} C_1 (t - 2C_2)}{2} \right] \tag{32}$$

Again condition $0 < \beta < 1$ ensures that the energy density is positive and that that bulk viscosity coefficient is real. The universe described by (27)-(32), for $C_2 \neq 0$, starts with finite energy density, cosmological term and bulk viscosity. During its evolution the energy density, cosmological term and bulk viscosity after the intial increase, tend to a constant value as $t \to \infty$.

4 Conclusions

In this paper we have considered Einstein's equations for the viscous flat FRW universe within the framework of Eckart theory and EIT with a variable cosmological term ($\Lambda \sim H^2$). Exact solutions of the field equations are obtained for a bulk viscosity index $m = 1/2$. It is found that in Eckart theory the cosmological term Λ varies as the inverse square of time, which matches its natural dimension. In EIT, for $m = 1/2$, the general solution is obtained in parametric form. A de Sitter solution given by eq. (11) exists for $m \neq 1/2$ in both the cases. For $m = 0$ and $\gamma = 0$, in Eckart theory, there is a period of hypherinflation, during which the cosmological term grows enormously. Some other solutions for the particular choice of parameters are also obtained and their evolution is discussed. It can be noted that in the limit $\beta \to 0$ the evolution equation of the Hubble factor H, in both cases, reduces to the one with no cosmological term.

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