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An Exploration of Final Year, Pre-Service Mathematics Teachers' Learning Experiences in Solving Non-Routine Financial Mathematics Problems at a South African University

By

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DECLARATION BY CANDIDATE

I, Tatolo Talasi, hereby declare that this thesis, submitted for the qualification of Doctor of Education (D.Ed.) in the Department of Mathematics, Science, and Technology Education, Faculty of Education, at the University of Zululand, has never been submitted for any degree, or any examination, to this, or any other, university. I also declare that it is my own unaided work, supervised to completion by Dr Anilkumar Krishnannair. I further declare that all sources used in compiling this thesis have been acknowledged.

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ABSTRACT

This study explored pre-service mathematics teachers (PSMTs) learning experiences in solving non-routine mathematical problems in financial mathematics. To this end, the theories that formed the foundation of this study were the realistic mathematics education, socio-constructivist theory of problem solving and mathematical proficiency.

Participants were drawn from an initial teacher education programme in South Africa. They were in the final year of their four-year Bachelor of Education degree, specializing in Mathematics, Science and Technology Education for intermediate and senior phase teaching. A total of 11 pre-service mathematics teachers participated in the study.

The study employed a qualitative approach and a case study design. A single case study with multiple units was deemed appropriate for this investigation. The data for this investigation were generated through questionnaires, documents, observations and interviews. Firstly, PSMTs responded to an open-ended questionnaire which had 27 items. Secondly, PSMTs solved four non-routine mathematical problems, based on financial mathematics, in groups, over four weeks. The problem-solving sessions were conducted concurrently in different venues. The D_group and the M_group were video recorded during each session, while the T_group was audio recorded. Thirdly, and finally, the researcher interviewed the available members of each group.

Content analysis was used to interrogate the data from the various instruments. The results were presented in the form of tables and excerpts. The findings across the various instruments showed that the PSMTs used numerical strategy, pictorial strategy, verbal strategy, and algebraic strategy to solve mathematical problems in financial mathematics. None of the groups used graphical strategy. The PSMTs grappled with contextual, linguistic, representational and transformational activities of algebra in their search for solutions to non-routine mathematical problems in financial mathematics. However, all the groups seemed to lack a knowledge of the time value of money principle and had inadequate knowledge of mathematical language. Consequently, all the groups committed linguistic and contextual errors, more than mathematical errors.

It is recommended that LTSMs' authors, curriculum developers and mathematics education instructors and lecturers design and implement learning outcomes which extend beyond the mere acquisition of mathematical concepts and mathematical skills. The learning outcomes that foster the acquisition of mathematical language, particularly financial mathematics language, should be prioritize in the assessment activities of PSMTs. Concerted efforts should be made to explicitly educate and expose PSMTs to problem situations that require the use of graphical problem-solving strategies, which have seemed missing in this cohort of PSMTs. The teaching of financial mathematics should focus upon the use of the time value of money principle and compounding of interest.

To maximize the learning experiences of pre-service mathematics teachers (PSMTs) in solving problems in financial mathematics, mathematics education lecturers may have to supplement the examples and practice problems given in PSMTs textbooks, and other LTSMs, to include non-routine mathematical problems, which focus upon the use of graphical problem-solving strategy that seemed to be highly underdeveloped in this cohort of PSMTs. It is further concluded that mathematics education lecturers should engage PSMTs with non-routine mathematical problems in financial mathematics with superfluous (and or missing) information and in higher cognitive demands to maximize PSMTs' learning experiences in solving these problems which are rarely found in textbooks and other LTSMs.

Key words: Algebraic activities, contextual activities, financial mathematics, learning mathematician, linguistic activities, non-routine problem, teaching mathematician, problem-solving strategies.

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DEDICATION

To my daughters, Hlompho Hilda Talasi and Keitumetse Talasi.

To my late parents, Mafusi Alice Talasi and Makhabane Ntjellane Talasi.

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ABBREVIATIONS

AB	Account balance
CAPS	Curriculum and Assessment Policy Statement
DBE	Department of Basic Education
DOE	Department of Education
FM	Financial mathematics
ITE	Initial teacher education
LTSM	Learning and teaching support material
PSMT	Pre-service mathematics teacher
OECD	Organization for Economic Co-operation and Development

CHAPTER ONE: BACKGROUND

1.1 Context

This study has been conducted against the backdrop of the South African schooling system. There are nine provincial departments of education in South Africa, operating under the national Department of Basic Education (DBE). Each provincial department of education is responsible for implementing the national educational policies and monitoring and evaluating educational progress within their province, while the DBE is responsible for designing policies that guide schooling practices (Feza, 2014a). The schooling system is divided into four phases, namely the Foundation Phase (FP), consisting of Grade R to Grade 3, Intermediate Phase (IP), which is made up of Grade 4 to Grade 6, Senior Phase (SP), which incorporates Grades 7 – 9, and the Further Education and Training (FET) Phase which is composed of Grade 10 to Grade 12.

1.2 Types of Schooling

South Africa has two types of schools. Firstly, it has private (independent) schools whose examinations are governed by the Independent Examination Board (IEB) and are characterized by comparatively high performance in international assessments, such TIMMS 2011 and TIMMS 2015. Reddy et al. (2016) pointed out that 81% of learners from independent schools who partook in TIMMS 2015 performed at, or above, the low benchmark of 400 points, compared to 19% of learners in no fee public schools and 60% of learners in fee paying public schools. Independent schools are generally attended by learners from wealthy families as they charge relatively high fees.

The second category of schools is the public schools which sit for the National Senior Certificate (NSC) examination controlled by the Department of Basic Education through Umalusi, as its examining and certifying body. Public schools are placed into five quintiles (see Chapter 2, section 2.4), depending on the socio-economic status of the population in which the school is located. Quintile 1, 2 and 3 schools are found in the poorest communities and get 100% subsidy from the government. However, schools in these quintiles are generally the most underperforming public schools in the country (see Chapter 2, section 2.4). Quintile 1, 2 and 3 schools are fee free schools and have the National School Nutrition Programme (NSNP). Quintile 5 schools are the richest of the public schools and get the least subsidy from the government and they charge fees. They are relatively the highest performing public schools.

1.3 Educational Dilemma

The common problem across all schools in South Africa is learners' relatively low performance in mathematics, as reflected by international assessments, such as TIMMS (Reddy et al., 2013; van der Berg et al., 2011). Learners' underachievement in a critical subject like mathematics has prompted all stakeholders to ponder a solution to the national crises. In South Africa, numerous curriculum changes and teacher training initiatives aimed at improving learners' performance in mathematics have been instituted and, in most cases, concluded with almost no impact on performance. Despite massive investments of resources in the teaching of mathematics, South African learners' performance in this subject is far below the international average as seen in international assessments (Reddy et al., 2013; van der Berg et al., 2011). Huge variations in educational performance in mathematics across schools and contexts indicate that South Africa's legacy of educational inequality is continuing and mirrors the country's societal socio-economic inequalities (Reddy et al., 2016). The majority of school leavers enter higher education, in particular teacher education, with huge educational gaps in subjects like mathematics. It is therefore important for teacher education to bridge these educational gaps, and so avoid perpetuating societal inequalities, while innovating in line with the South African National Development Plan 2030 (NDP, 2030) and revolutionizing teacher education to meet the demands of the Fourth Industrial Revolution (4IR).

Revolutionizing mathematics education implies teaching mathematics with problem solving in mind. The mathematics classroom should be a haven for nurturing intuitive and wild problem-solving strategies. This researcher believes that problem solving at school would only be accelerated by allowing prospective teachers to experience and solve 'messy' problems during their training and thus build a flexible knowledge of mathematics that goes beyond the 'regurgitation' of tacitly learnt algorithms which make no sense. It is against this framework that this researcher explored pre-service mathematics teachers' learning experiences in solving non-routine mathematical problems in financial mathematics.

The causes of poor performance in mathematics are diverse and vary across both socio-economic status, groups and schools. Mugisha (2012) notes that mathematical anxiety (fear of mathematics) negatively affects performance in mathematics. The poor teaching of mathematics, the lack of teaching resources, such textbooks, quality of mathematics teachers in a school, home background, lack of management at school and the functionality of a school have been cited as contributors to underperformance (Spaull & Kotze, 2015; van der Berg et al., 2011). However, Maree, Aldous, Hattingh, Swanepoel, and Van der Linde (2006) found that at the heart of this relatively low performance in mathematics is South African learners' inability to deal with questions set at a problem-solving level in international assessments. Mugisha (2012) argued that problem solving is the most important cognitive activity needed both at school and workplace. Exploring learning experiences of PSMTs' problem solving abilities as they exit teacher education would give valuable information about the mathematical experiences and abilities of teachers entering the teaching profession. Such information may be used in revamping and aligning ITE programmes with national and international aims and ways of teaching mathematics to enhance PSMTs' problem-solving abilities or in providing in-service interventions aimed at empowering teachers to implement the national curriculum holistically.

1.4 Teacher Education in South Africa

Previously, teacher education in South Africa was conducted in racially segregated colleges of education, each with its own curriculum aimed at serving a narrowly-defined and perceived role of that race-group in the apartheid society (Spaull, 2013a). Since the advent of democracy in 1994, teacher education is offered by universities working in silos to achieve national goals.

Teacher training has been receiving attention for various reasons. Adler (2005) found that teacher education programmes emphasized mathematics content more than mathematics education. The quality of teachers, and their level of readiness to enter the teaching profession, has also been questioned. South African Grade 6 teachers, for example, seemed to have an inadequate knowledge of the mathematics they are teaching (van der Berg et al., 2011). Spaull (2013b) added that mathematical knowledge of teachers in rural schools is substantially lower relative to other South African teachers. Hence, there is a need to continually examine and benchmark pre-service mathematics teachers' learning experiences to ensure the local and global competitiveness of teachers, especially their problem-solving experiences.

Prior to the inception of Curriculum 2005 (C2005), financial mathematics constituted a very small proportion of the school curriculum and it was only available to learning mathematicians[1] taking mathematics on Standard Grade. Even then only 'simple interest' and 'compound interest' were taught. When financial mathematics was extended to all grades, beginning with simple every day transactions at a primary school level, culminating in the teaching of annuities in Grade 12, most teachers did not have a working knowledge of the topic from school, as learning mathematicians, or from teacher education, as prospective teachers, but they were expected to provide quality teaching of this critical aspect of mathematics.

It seems essential to find the extent to which current pre-service mathematic teachers (PSMTs) have been skilled in solving non-routine problems in financial mathematics.

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[1] The term 'learning mathematician' refers to anyone who is enrolled to learn mathematics either at a school, college or university outside the faculty of education. This researcher uses this term to incorporate pupils, learners, students, and scholars, as these terms are used contextually in different countries. For example, in South Africa, a learner is anyone who is enrolled in a school, while a student refers to anyone enrolled at a college or university level.

1.5 Rationale

This researcher has observed over the years that most high school learning mathematicians and first year pre-service mathematics teachers[2] (PSMTs) do well when given mathematical problems such as $factorise : 2x^2 + 5x + 3$. However, there is relative underperformance when they are confronted with problems such as:

> *There are n points, no three of which lie on the same line. How many straight line segments are needed to connect every possible pair?*

Most learning mathematicians fail to figure out how to respond to problems of this nature. Similar observations have been made by Olanoff, Lo, and Tobias (2014, p. 5) who indicated that "pre-service teachers had a strong procedural knowledge of fractions but lacked understanding of the meanings behind the procedures or why the procedure worked".

Loji (2012) posited that Year 1 learning mathematicians enrolled for the module 'Engineering 1' lacked problem-solving skills which made it difficult for them to manoeuvre through qualitative problems, which do not hold any indications as to which steps or procedures could be used to solve the problem. In trying to find an explanation, the researcher looked at the South African school curriculum to find out which teaching approach is being advocated for in mathematics. The Department of Basic Education (2011b) (DBE) has asserted that mathematics teaching is aimed at developing problem-solving and cognitive skills. It further pointed out that teaching should not be limited to "how" but should rather feature the "when" and "why" of problem types and that learning procedures and proofs, without a good understanding of why they are important, will leave learning mathematicians ill-equipped to use their knowledge in later life.

The DBE advocates problem solving in the teaching of mathematics but the products of this school curriculum seem to be procedurally fluent and lack problem-solving strategies. OECD (2014) argued that modern life is all about problem solving and that the changes in society, environment and technology imply that the way mathematics is used is rapidly evolving. OECD further posited that education is not immune to these changes. Teachers

[2] Pre-service mathematics teachers refers to anyone learning mathematics with the intention of teaching it.

are now expected to teach in ways that will enable learning mathematicians to use acquired knowledge and skills in more sophisticated ways than before. The time for teaching routine procedures has changed to equipping learning mathematicians with complex skills (i.e. higher order thinking skills (HOTS)) so that they are in a better position to make mathematically informed decisions, both at work and in their everyday life (OECD, 2014).

Marchis (2012) made similar observations and reported that Romanian learning mathematicians perform above international average on routine[3] problems but score lower than international average on non-routine [4] mathematical problems. Mullis, Martin, Foy, and Arora (2012) argued that teacher training and preparation is critical to learner performance and problem-solving abilities.

Initial teacher education (ITE) programmes in South Africa have been under scrutiny since the advent of democracy for the under preparedness of mathematics' teachers. Adler and Davis (2006) argued that mathematics' courses designed for teachers across South African universities emphasize the compression of mathematical ideas instead of "unpacking" or decompression, which is required for teaching mathematics. They posited that the mathematical knowledge which teachers need to know and know how to use in their practice is specific to teaching and is different from the mathematical knowledge used in other professions. Given that teachers are custodians of teaching and learning and are at the forefront in preparing learning mathematicians for various future endeavours, it is essential to explore learning experiences which pre-service mathematics teachers bring into the schooling system, as these learning experiences may restrict or accelerate non-routine mathematical problem-solving abilities of learning mathematicians.

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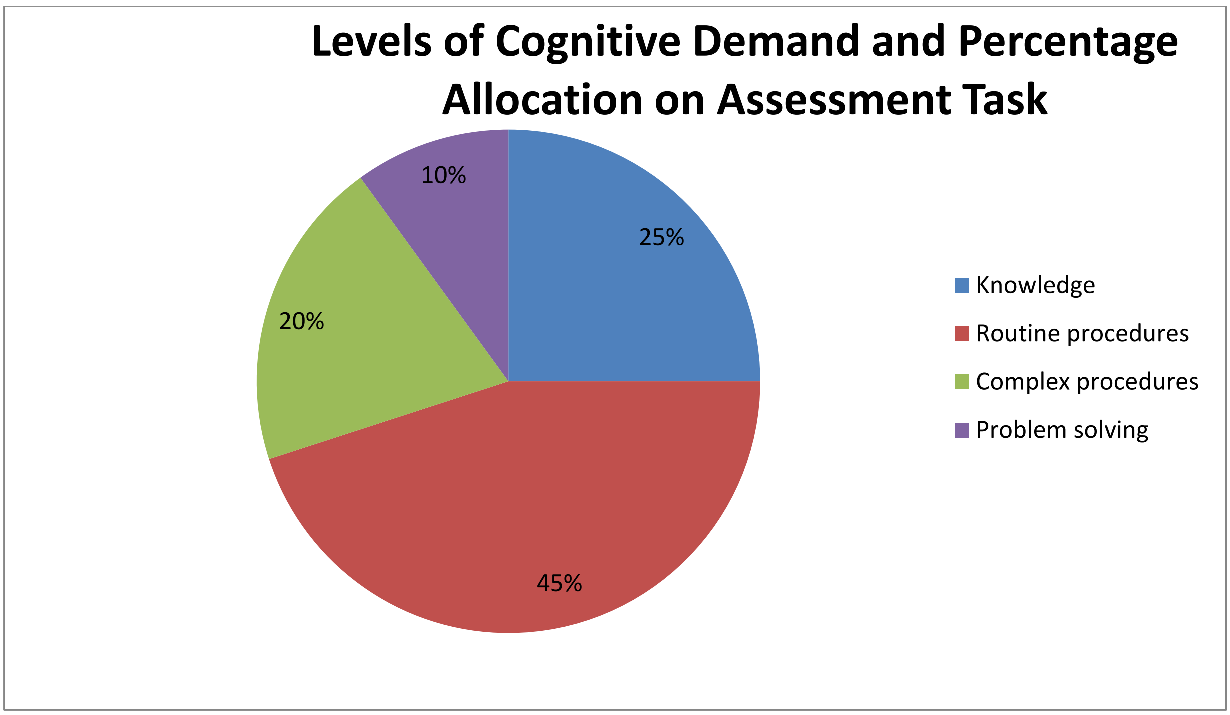
[3] Routine problems are problems with a clear route to the answer and require the repetition of a learned procedure or algorithm to be solved.

[4] Non-routine problems are problems which do not hold any clue of how they could be solved. They require one to develop, execute and evaluate a plan to solve them (Aljaberi & Gheith, 2016).

1.6 Problem Solving within the South African National Curriculum: A Brief Introduction

The National Curriculum and Assessment Policy Statement, commonly known as Curriculum and Assessment Policy Statement (CAPS) (Department of Basic Education, 2011b) revealed certain interesting aspects about problem solving. The section on assessment shows four levels of cognitive demands and their corresponding weightings in percentages used when examining learning mathematicians. Ironically, considering the great need for problem solving, it has the lowest weighting (see figure 1).

Figure 1: Levels of Cognitive Demand and Percentage Allocation on Assessment Tasks



Source: Adapted from CAPS Grades 7-9 (Department of Basic Education, 2011b)

Figure 1 shows four cognitive skills which must be demonstrated by learning mathematicians in the General Education and Training (GET) phase. 'Routine procedure' takes the largest share of assessment marks, followed by 'knowledge' and 'complex procedures' at 25% and 20% respectively. Mathematical tasks that can be solved using routine procedure place less cognitive demand on the problem solver in that one has to recall a learnt fact, formula or algorithm to get the answer. Those types of tasks are well structured and are mainly found in school textbooks, e.g. $solve\ for\ x\ in\ x^2 + 4x + 4 = 0$. Tasks such as those are often given after learning mathematicians have been taught a procedure; as such they do not afford learning mathematicians an opportunity to use their informal problem-solving strategies or to see mathematics as sensible and worthwhile. This researcher is of the view that routine tasks create passive learning mathematicians who believe that only teachers' methods are correct, since routine tasks often limit individuals' creativity. Hence there is a need to greatly diversify learning activities.

Although problem solving is allocated only 10% weight in school mathematics assessment, it is the most frequently used cognitive skill beyond the school. In support, OECD (2014) has argued that in the workplace, the move towards problem solving has been necessitate by the arrival of computers and computerized machines which could perform all routine tasks in a few minutes. OECD (2014) added that workers are now required to deal with more complex, unexpected and unfamiliar situations to get the best out of the computerized machines which work alongside them. Teachers' classroom practices are not exempted from technological advancements. Hence, there is a shift towards educating PMSTs in such a way that they could go beyond a mastery of facts and procedures to empowering them to be able to think creatively when confronted by a complex, non-routine mathematical problem and to persevere in searching for a solution where the effectiveness of their intervention is not predictable.

It seems that problem solving, while being advocated within the South African school curriculum, is not a highly prioritized practice, at least in the assessment of learning mathematicians' attainments. Perhaps this is an example of what Adler (2005) termed 'systemic problems', which, in this case, is that an assessment does not align with what the curriculum advocates. The small percentage of problem solving in assessment tasks indicated by the CAPS document may imply that the school curriculum does not match what teachers know both from school (as previous learning mathematicians) and from teacher education, i.e. teaching and assessing problem solving.

Mogari and Chirove (2017, p. 5) pointed out that problem solving is a higher order thinking skill (HOTS) which requires learning mathematicians to "*synthesize and coordinate knowledge and skills and apply them on novel problem situations*". In agreement, Loji

(2012) postulated that problem solving encompasses synthesis, analysis and evaluation. However, figure 1 shows that the South African school curriculum, in its current form, focuses on low-order thinking skills, whereas international assessments, such as TIMSS (Trends in International Mathematics and Science Study), and PISA (Programme for international Students Assessment) and research on mathematics education demand the use of HOTS, in particular problem solving in the teaching and learning of mathematics.

School leavers lacking in problem-solving skills are likely to struggle in other environments, such as their workplace, daily living activities and in pursuing further studies at universities (Loji, 2012; Nieuwoudt, 2015a; OECD, 2014), which demand the use of HOTS. In mathematics, lower-order thinking skills are mainly rule-based. Kilpatrick, Swafford, and Findell (2001, p. 259) argued that rule-based instructions do not afford learning mathematicians opportunities to create meaning or learn why, when and how, to apply the learnt algorithms and procedures which constitute the routinized learning that can lead to "*forgetting, unsystematic errors, reliance on visual cues and poor strategic decisions*".

According to Nieuwoudt (2015a) and Department of Basic Education (2011b, p. 8), the aim of teaching mathematics in South Africa is threefold. Firstly, mathematics teaching is meant to "***enhance learner's problem solving skills*** essential for their everyday living", secondly to attain one of the general aims of the South African curriculum, namely *to "**identify and solve problems and make decisions using critical and creative thinking**"* and thirdly, to accomplish one of the specific aims of the South African curriculum which is to "***apply mathematics to solve problems, using acquired knowledge and skills***". The bold text seems to advocate teaching mathematics through problem solving since learning mathematicians are to identify and solve problems. The question is how this aim will be realized, since problem solving is allocated only 10% in assessment tasks.

Nieuwoudt (2015a) studied Grade 4 learning mathematicians as they developed and used mathematical models to solve non-routine mathematical problems. These 4^{th} graders solved non-routine mathematical problems by *drawing diagrams*, *counting*, *performing calculations and acting out some problems (italics added for emphasis)*. She concluded that the success in problem solving at a primary school level is dependent on the teacher's competence, dedication and ability to mediate learning.

Sepeng and Kunene (2015) conducted a study with Grade 6 learning mathematicians to determine which strategies are used by these learning mathematicians in responding to word problems. They concluded that although most learning mathematicians were overwhelmed by the linguistic demands of the task, *reading instructions aloud* and *explaining key mathematical concepts* were the most used strategies (italics added for emphasis). Näveri, Pehkonen, Hannula, Laine, and Heinilä (2011) warned that word problems are not automatically synonymous with problem-solving tasks as these may have a clear route to the answer. However, the findings by Sepeng and Kunene (2015) are relevant to this study as they show how learning mathematicians in primary schools solve unfamiliar problems and such strategies maybe used by pre-service teachers.

Gaigher, Rogan, and Braun (2006) conducted a study with 16 high schools to investigate the effect of structured problem-solving strategies on performance and conceptual understanding. They concluded that learner performance was enhanced by teaching through problem solving. This research demonstrated that a lot of research has been conducted at school level, both on problem solving and problem-solving strategies, but there seems to be gap in the literature on pre-service teachers' mathematical problem-solving strategies at university level in South Africa, although these teachers are expected to engage learning mathematicians in non-routine mathematical problem solving.

1.7 Learner Performance in School Mathematics: Need for Reconceptualising PSMTs' Learning Experiences in South Africa

Van der Berg et al. (2011) argued that, despite the numerous reforms in education and curriculum modifications, the quality of education has remained disappointing in South Africa. They posited that most learning mathematicians are performing far below the curriculum standards.

Expatiating on this point, (Spaull, 2013b) reported that Grade 6 mathematics teachers lacked mathematical knowledge in addition to problem-solving skills since they could not solve the same problems the curriculum expected their learning mathematicians to solve.

Bansilal, James, and Naidoo (2010) argued that learning mathematicians' weak mathematical skills are an indication of a low education quality, in general. They posited that the education quality, especially in mathematics, can be improved if teachers use effective problem-solving strategies which build on learning mathematicians' current mathematical knowledge. The poor performance of learning mathematicians as indicated by both national and international assessment, coupled with the weak subject matter knowledge and the lack of problem-solving skills of in-service teachers have prompted the government to initiate in-service upskilling programmes to empower teachers (Spaull, 2013a). Usually, these initiatives target subject matter knowledge, i.e. mathematics content, instead of the teachers' pedagogical content knowledge, i.e. knowledge of how mathematics is taught and learnt and are only available for in-service teachers. This implies that if PSMTs exit ITE programmes without having been skilled in teaching mathematics through problem solving, some aims of the South African school curriculum would not be achieved and South African learners would continue to perform poorly in assessments which prioritize HOTS, particularly problem solving. Sepeng and Kunene (2015) argued that development activities should focus on assisting teachers on how to teach mathematical word problems in the early grades using learning mathematician's everyday situations.

These research findings have not only prompted institutions that offer initial teacher education (ITE) programmes to continually review, evaluate and update their programmes to include new innovations in preparing teachers but have also prompted the South African Department of Higher Education (DHE) to unifying initial teacher education (ITE) programmes, at least at policy level, by introducing Minimum Requirements for Teacher Education Qualifications (MRTEQ) framework. Critical to this document is an exit level outcome which demands that teachers should demonstrate self-directedness and originality in problem solving (Department of Higher Education, 2015).

Teaching mathematics through non-routine mathematical problem solving is a new orientation, very often absent from the South African teacher's repertoire. Mogari (2014) attributed this absence to examination driven teaching in most South African schools, which emphasize rote learning and memorization. Mogari further pointed out that teachers who participated in an in-service programme on the ethno-mathematics approach did not embrace the approach in their teaching but when requested, they used it well. It seems that for teaching innovations to be implemented in South African classrooms they have to be reflected in learning mathematicians assessment tasks. Loji (2012) asserted that most South African learning mathematicians entering higher education lack problem-solving skills and this jeopardizes their progress to second year. Aligning herself with Loji, Nieuwoudt (2015a) drew on Trends in International Mathematics and Science Study (TIMSS) to argue that the poor performance of South African learning mathematicians in international assessments emanates from an inadequate acquisition of concepts and mathematical skills, particularly problem-solving skills.

The South African School curriculum aims to produce learning mathematicians who are *able to* "identify and solve problems and make decisions using critical and creative thinking" (Department of Basic Education, 2011b, p. 5). This is what mathematicians do: identify and solve problems. Thus, the curriculum accentuates learning mathematics to doing mathematics, of which solving everyday life [non-routine] problems is a critical component (Fauzan, 2002). This aim aligns with modern developments in teaching and learning mathematics through problem solving.

Aljaberi and Gheith (2016) argued that problem solving develops and improves logical thinking as well as decision-making skills. Hence, teachers should organize activities that encourage learning mathematicians to make decisions and to reason as they learn mathematics for them to make sense and reinvent mathematics and problem-solving strategies through active participation and collaboration. Problem solving helps to strengthen both the content knowledge and pedagogical knowledge as it affords problem solvers time to reflect on the problem-solving process, the outcomes and also to communicate their problem-solving strategies. In support, Abdullah, Ibrahim, Surif, Ali, and Hamzah (2014b) asserted that when engaged in problem solving, learning mathematicians experience the power and usefulness of mathematics. Non-routine problem solving has a built-in uncertainty since it does not always lead to an answer and the problem is open to being solved with different approaches. For teachers to embrace uncertainty, they should have sound conceptual understanding of mathematics and mathematical pedagogic knowledge.

There is consensus among researchers in mathematics education (Karatas & Baki, 2013; Marchis, 2012; Nieuwoudt, 2015a) that problem solving is critical to the teaching and learning of mathematics at school. Learning mathematicians who solve non-routine mathematical problems tend to attain better scores in tests, national assessments and international assessments, such as TIMSS and PISA (Grønmo et al., 2015; Mullis et al., 2012; OECD, 2014).

Abdullah et al. (2014b) added that for learning mathematicians to attain better scores in assessments, teaching and learning should involve problem-solving skills that are robust, covering the whole curriculum. Engaging learning mathematicians in non-routine mathematical problem solving allows them to attain mathematical proficiency since their previous learning experiences and contributions are discussed, interrogated and integrated into the new learning (Barnes, 2004; Gravemeijer & Doorman, 1999; Kilpatrick et al., 2001). The question that arose for this study is: in which problem-solving strategies are pre-service mathematics teachers skilled as they exit ITE programme?

1.8 Statement of the Problem

Problem solving is the most sought after skill in the workplace and in everyday life activities (OECD, 2014). However, most South African learning mathematicians and in-service teachers seem to lack this skill. As a result, both learning mathematicians and in-service teachers perform poorly on problems which are on a problem-solving level (Spaull, 2013b). Despite this revelation, there is a dearth in South African literature explicating the particular types of problem-solving strategies and algebraic activities in which pre-service mathematics teachers should be skilled as they exit ITE programmes.

Further, most studies done regarding mathematical problems-solving strategies in South Africa have focused on learning mathematicians but few on in-service teachers (Mogari

& Chirove, 2017; Mogari & Lupahla, 2013; Sepeng, 2014; Sepeng & Kunene, 2015; Spaull, 2013b). Hence, the problem-solving abilities of teachers, especially pre-service mathematics teachers remain under researched in South Africa.

The Department of Basic Education (2011b) has contended that learning mathematicians in Grades 7 and 8 are not expected to use mathematical formulae when solving problems in financial mathematics. However, the DBE does not specify the strategies that should be used in these grades. Despite the DBE's contention, Da Silva, Pournara, and Mafuya (2009) posited that teaching financial mathematics in schools tends to be dominated by substituting values in a formulae. While this approach is usually favoured by most textbooks, Bostock and Chandler (2015, p. 440) stated that:

> *Although there are rules/**strategies** to solve a few particular forms of problems, in most cases such rules/**strategies** do not exist. **Solving non-routine problems** is an art; experience will suggest a likely **strategy** of solving the problem without guaranteeing a solution.* (bold added for emphasis)

The question for this study has been if teachers are to teach mathematics without formulae or standardized algorithms, to what kind of learning experiences would pre-service mathematics teachers be exposed? Problem solving, as the most significant aspect of mathematical learning, remains under-emphasized at the level of classroom practice. Hence, it is important that pre-service mathematics teachers' faculties of problem solving, that are their current learning experiences, are explored. To do this, this researcher studied problem-solving strategies and algebraic activities that groups of PSMTs deploy in solving non-routine problems in financial mathematics.

The study conducted by Khalo and Bayaga (2014) on Grade 10 mathematical literacy learning mathematicians solving problems on financial mathematics showed that the 10th graders made numerous errors and they recommended that error analysis be incorporated into ITE programmes to alleviate the problems experienced by learning mathematicians. After studying Grade 12 learning mathematicians solving financial mathematics problems which demanded the use of multiple representations, Da Silva et al. (2009) concluded that non-routine financial mathematics problems offer more information about learning mathematician's mathematical understanding and skills than the 'learnt by heart' procedures that involve substituting values into formulae.

Makonye (2017) posited that pre-service teachers who partook in his study showed a weak conceptual understanding of financial mathematics concepts (nominal and effective interest rates) as their explanations were based on the formula and technology used (calculators in this case).

Pournara (2014), working on mathematical knowledge for teaching (MKfT), studied pre-service teachers solving problems on annuities. He concluded that the knowledge required for teaching financial mathematics entails mathematical knowledge, pedagogical knowledge and contextual knowledge of banking. However, he did not indicate what learning experiences, specifically problem-solving strategies, teachers need to know and be able to use in solving financial mathematical problems. Other authors concurred with the DBE that learning mathematics should enable learning mathematicians to identify and solve problems, although they differed with each other on what should constitute PSMTs' learning experiences necessary for realizing this educational goal especially in the teaching and learning of financial mathematics (Khalo & Bayaga, 2014; Makonye, 2017; Pournara, 2014).

The studies cited above show that much research has been conducted on financial mathematics in South Africa. However, there is virtually no study in South Africa which focused on the strategies and algebraic activities used by pre-service mathematics teachers in solving non-routine financial mathematics problems. Hence, there is a need to explore the learning experiences of the PSMTs who would be teaching this topic to establish the level and extent of their problem-solving abilities.

1.9 Purpose of the Study

The purpose of this study was to explore pre-service mathematics teachers' learning experiences with a special emphasis on the mathematical problem-solving strategies used by this cohort of pre-service mathematics teachers as they solved non-routine mathematical problems. The study further sought to establish pre-service teachers' espoused conceptions and understanding of problem solving, as well as the algebraic activities that they draw on during problem solving. The study explored the learning experiences of pre-service mathematics teachers in the context of such teachers solving non-routine mathematical problems in financial mathematics. The study culminated in a set of recommendations aimed at curricular and instructional improvement of teacher training programmes.

The sections which follow deal with the objectives and research questions guiding the study, as well as the rationale for choosing these questions research.

1.10 Objective of the Study

The objectives of the study were:

- To explore the mathematical problem-solving strategies and algebraic activities used by pre-service mathematics teachers when solving non-routine problems.
- To explore problem-solving strategy flexibility within and across problems as demonstrated by pre-service mathematics teachers.
- To determine pre-service mathematics teachers' espoused conceptions about problem solving.

1.11 Research Questions

1.11.1 Main research question

The following main research question underpinned this study:

How can the learning experiences of pre-service mathematics teachers be optimized in the use of strategies in solving non-routine problems in financial mathematics?

1.11.2 Critical research questions

The answer to the main research question will be formulated based on the answers to the following critical questions:

1. What are the cognitive strategies that pre-service mathematics teachers use in the context of their problem solving in financial mathematics?

2. To what extent do pre-service teachers exhibit intra-task or inter-task strategy flexibility?
3. What are pre-service mathematics teachers espoused conceptions concerning non-routine mathematical problem solving?
4. To what extent do pre-service teachers draw on algebraic activities when solving non-routine mathematical problems?

1.12 Brief Explanation of the Research Questions

The main research question is formulated according to the overarching aim of this study. It focuses on the prospects of enhancing the quality of the learning experiences of pre-service teachers in relation to their exposure to teaching, learning and the content of financial mathematics. At a more specific level, the research has focused on the problem-solving strategies which are subsumed under the larger idea of the learning experiences. An emphasis on the problem-solving strategies is justified owing to the critical role the mastery of such strategies plays in enhancing the quality of learning experiences in general. This main question will therefore be answered by the answers to the critical research questions.

Critical questions one through three are about the mathematical problem-solving strategies which will be used by final year pre-service mathematics teachers and the mathematical resources (algebraic activities) that will be deployed. These questions are important because to understand how pre-service mathematics teachers solve non-routine mathematical problems in financial mathematics, their problem-solving strategies must be investigated. Furthermore, for teachers to embrace problem solving in their teaching, they must be competent problem solvers and should know which problem-solving strategies are available to them and their learning mathematicians (Posamentier & Krulik, 2008).

In a similar manner, for teacher's classroom practices to enhance learning mathematician's learning of mathematics, teachers must have a repertoire of mathematical problem strategies to apply when solving mathematical problems. In South Africa, classroom practices in learning mathematics are frequently dictated by the teacher. This warrants that teacher training institutions frequently examine PSMTs' competencies in problem-solving strategies to assist them with up-to-date, efficient and effective practices necessary for implementing the school curriculum holistically. Problem-solving strategies are critical to learning mathematicians' mathematical reinvention (Barnes, 2004). Such strategies are deemed essential in workplaces and for everyday living since the problems encountered in these environments usually demand HOT skills.

Therefore, it is important to examine both the problem-solving strategies and algebraic activities which are likely to be used by these PSMTs to legitimize their actions as they solve mathematical problems, as well as their espoused problem-solving conceptions which could influence their classroom practices.

1.13 Significance of the Study

This study sought to explore prospective[5] teachers' learning experiences with special reference to their mathematical problem-solving strategies, which they use in solving non-routine mathematical problems, and the algebraic activities which they draw on during problem solving. Olanoff et al. (2014) argued that to design mathematics education courses which can help prospective teachers develop a robust understanding of mathematics in general and *mathematical problem solving* (italics added for emphasis), one needs to start by determining what prospective teachers know. If teachers possess a wealth of problem-solving strategies, they are more likely to embrace problem-solving activities during teaching, and thus achieving the goal of education (Abdullah et al., 2014b).

It was anticipated that the study would reveal which problem-solving strategies are commonly or rarely used by prospective teachers, and thus indicate to curriculum

[5] In this study, the terms 'pre-service teacher' and 'prospective teacher' will be used interchangeably.

designers, lecturers and subject advisors which strategies need emphasis during teacher training or interventions.

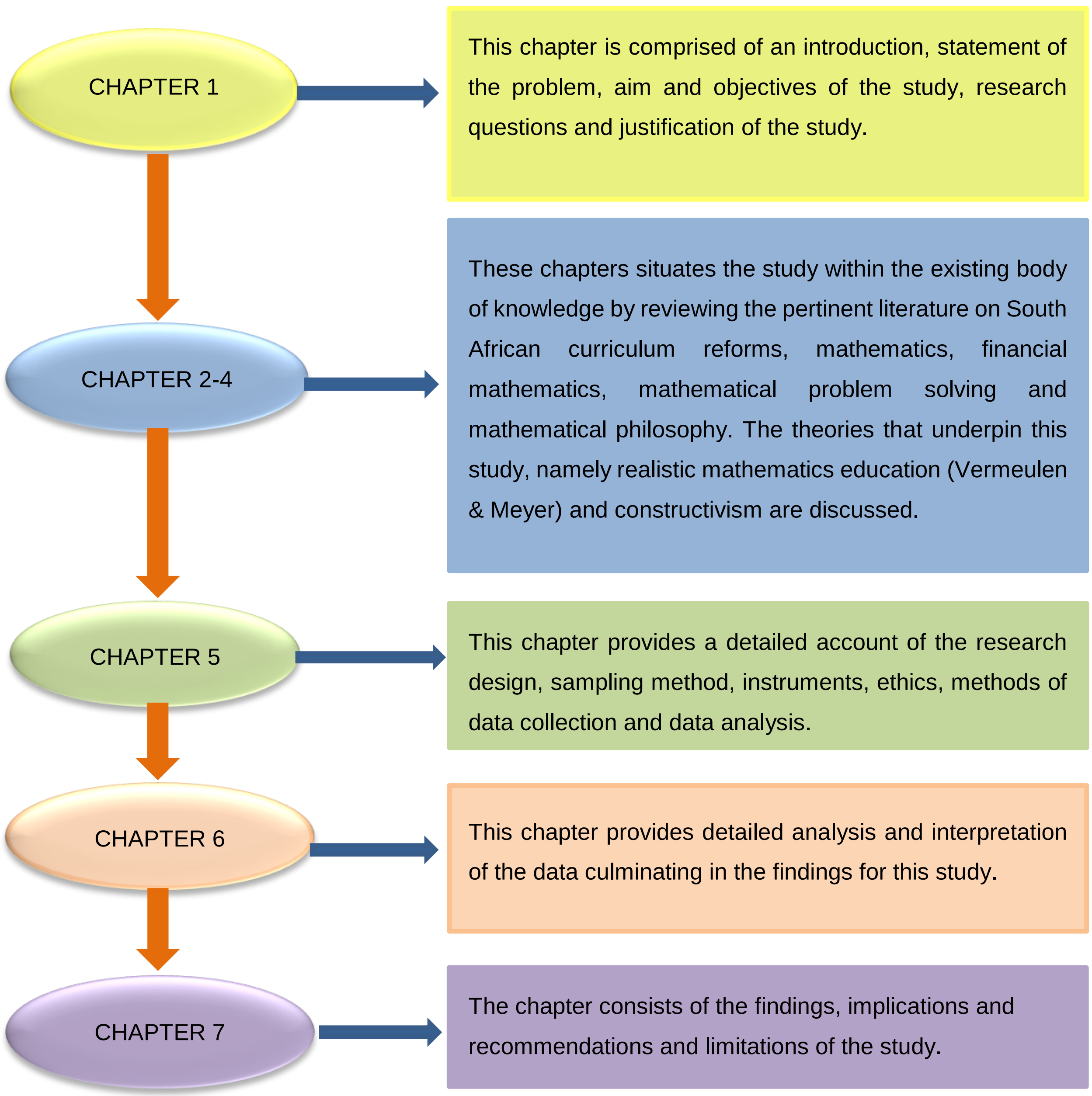
Bishaw (2011) asserted that the teacher factor is critical to the implementation of educational innovations. In agreement, Bulut and Karamık (2015) argued that a teacher should have a repertoire of problem-solving strategies for a problem and know when and how to use them. Thus, the major concern of this study was to critically explore the problem-solving strategies used by pre-service teachers and the algebraic activities they draw on when engaging in mathematical problem solving with the intention to improve teacher education programmes.

It was further anticipated that the information and lessons from this study would add new ideas to the debate concerning the manner in which pre-service mathematics teachers should be trained at a university level to embrace problem solving in their teaching. Research in this area (problem-solving strategies) is extremely rare in South Africa at the university level, although this is where teachers are being trained. Previous studies on problem solving have only looked at problem solving or problem-solving strategies in isolation of the mathematical content. This study took mathematical content into consideration under algebraic activities. In this way, the study hoped to go beyond the availability or absence of strategies and, hence, show in which mathematical activity the strategies are lacking.

Unlike other faculties of education across South Africa, which require at least 50% in matric mathematics for registration, the institution where this study was conducted requires a minimum of only 40% in matric mathematics for a candidate to get enrolled into a Bachelor of Education programme, intended for training mathematics and science teacher. Hence, this study sought to illuminate the problem-solving experiences (particularly their mathematical problem-solving strategies) of the relatively under-prepared learning mathematicians enrolled into an initial teacher education programme.

This study was conducted at an institution that mostly serves historically disadvantaged communities. Due to its rural location, this university does not attract many school leavers who passed National Senior Certificate (NSC) examination with high marks as they opt for universities located in the cities or universities that are known for high academic excellence. Because of this, this study enabled this researcher and other stakeholders to compare the findings of this study with those made in other institutions operating under different conditions. Other institutions in similar situations may be able to draw on these findings to improve their programmes.

1.14 Structure of the Thesis



1.15 Summary

In this chapter, the South African schooling system has been discussed. The rationale for this study has been provided by drawing on the South African School curriculum, and this researcher's teaching experience, at school level, as well as in ITE programmes. Problem solving is not only an intended teaching strategy in South Africa but it is a practice in many other countries, such as Singapore, Japan, Netherlands, and South Korea, countries which perform well in international assessments, such as TIMMS. The research problem, research questions and the significance of this study have been expounded upon. The chapter was concluded by giving a brief outline of the thesis.

CHAPTER TWO: LITERATURE REVIEW

2.1 Introduction

In the sections that follow, the literature pertinent to this study is discussed in an effort to anchor the constructs that are critical in this study. These constructs are *learning experience; mathematics; problem solving; algebraic activities; problem-solving strategies; financial mathematics and espoused conceptions*. The study is located in the broad area of mathematics education with particular interest in pre-service mathematics teachers (PSMTs) problem-solving strategies in financial mathematics.

This researcher drew on curriculum reforms in South Africa; the history of mathematics teaching and learning in South Africa; South African learners' performance in local and international assessments, and research on mathematics education to substantiate the importance of this study. The researcher further drew on mathematical knowledge for teaching to highlight that mathematical problem-solving strategies are part of this knowledge (Kazima & Adler, 2006). This chapter concludes that the theoretical constructs which guide this study align with the philosophy underpinning the South African mathematics curriculum.

This researcher has noted that people study teaching after many years of peripheral participation in learning mathematics. This history does not only impact on what they learn as pre-service mathematics teachers but it is also critical to how they would teach mathematics. This view is also shared by Holm and Kajander (2012, p. 7) who posited that "Teachers are likely to continue to teach in the way they were taught if teacher educators do not find a way to interrupt this self-perpetuating cycle". It is this researcher's view that faculties of education should review and upgrade their ITE programmes to interrupt this cycle, by examining the learning experiences of their prospective teachers before they leave initial teacher education.

The teaching and learning of mathematics have been a point of contention for decades in South Africa, from both political and academic perspectives. The debates often centre around a number of issues, such as what mathematics should be taught at a particular grade, what skills and concepts should be included in the curriculum, and the quality of mathematics teachers. However, the question that is often neglected is what is the purpose of teaching mathematics? Previously, imparting computational skills and procedural fluency were prioritized (Barnes, 2004; Mogari, 2014). The arrival of pocket calculators and other technological devices which can carry out computations effectively and efficiently in a matter of seconds opened other mathematical avenues for teachers lest they become redundant.

This researcher is of the view that the primary goal of teaching mathematics is to inculcate in learning mathematicians the culture of inquiry and the independent problem-solving skills necessary for partaking in societal issues, as well as furthering their studies. To achieve this, PSMTs should be trained in ways that enable them to handle non-programmable mathematical skills, such a problem solving.

2.2 Learning Experience

In learning mathematics, learning experiences may consist of mathematical practices such as sense-making, observing, exploring, creating arguments using definitions and logical deductions (Szydlik et al., 2003). The Department of Basic Education (2011b) added that mathematics is based on observing, representing and investigating patterns and relationships. For Polya (2014), learning experience in mathematics consists of a four-step problem-solving heuristic. Krulik and Rudnick (1982) posited that problem-solving strategies are an essential component of PSMTs learning experiences. Kilpatrick et al. (2001) argued that algebraic activities are fundamental to PSMTs learning experiences. To solve problems in financial mathematics (FM), financial conventions and financial contextual factors become a necessary learning experience for PSMTs (Pournara, 2014, 2015b). In mathematics, the term learning experience has a bearing on different attributes of problem solvers'/PSMTs' knowledge needed for mathematical problem solving. These are an ability to use a variety of problem-solving strategies and heuristics, in-depth knowledge of mathematics and mathematical practices, as well as the knowledge of conventions and contextual factors across topics.

In this study, learning experience is defined as an interaction which arises and results in the formation of new knowledge, skill or behaviour or the modification of existing knowledge, skill or behaviour. The interaction has both the cognitive and social elements. Interaction may occur in planned settings, such as classrooms or in out-door environments, through studying, practicing, being taught, experiencing, discussion or reflection. In this study, PSMTs collaborate to solve non-routine problems in financial mathematics. In this way, their existing knowledge or problem-solving behaviour is modified or they learn new strategies of solving the problems. When PSMTs work together to resolve a problematic situation, their interactions are constrained and focused by the non-routine mathematical problem confronting them. They individually ponder about the ways of resolving the problem and share their thoughts with the other members. Their thinking depends on their prior exposure to everyday knowledge and previously learnt mathematics.

2.3 History of South African School Curriculum Reform in Brief

It is not the intention of this study to describe in detail the curriculum changes in South Africa or the curriculum processes of introduction, revision and implementation but it is useful to draw on curriculum history to show that curriculum affects the mathematics to be taught, how mathematics should be taught and that curriculum changes place different demands on the teachers' learning experiences. Unlike in most countries, curriculum reform in South Africa was politically driven, with the democratic government deciding on the content and the duration of mathematics lesson (Feza, 2014b; Mogari, 2014).

According to Mouton, Louw, and Strydom (2012), during apartheid, South Africa had 18 racially defined educational departments of education. They further pointed out that post 1994, these departments were reduced to nine provincial departments operating under one national department of education. South Africa had its first unified school curriculum implemented in 1998. The post-apartheid curriculum was called Curriculum 2005 (C2005), as its implementation would have reached Grade 9 by 2005. C2005 was underpinned by an outcome based educational philosophy. As a result of its educational philosophy, Curriculum 2005 was referred to as OBE by most teachers. In the new curriculum, the behaviours (outcomes) to be demonstrated by learners at the end of their learning were foregrounded.

According to Jansen (1998), C2005 was primarily a political response to the apartheid schooling system and had no modalities for changes in classrooms. Jansen’s views support this researcher’s view that changing the curriculum has not changed the way mathematics is taught in most previously disadvantaged schools. The South African School curriculum and most teachers’ classroom practices still aim at producing knowledge intensive graduates (knowledgeable in mathematical facts and procedures) at the expense of problem solving.

C2005 was a political response intended to purge the apartheid curriculum in schools, to introduce continuous assessment and to install OBE in schools (Jansen, 1998). Froneman, Plotz, Benadé, and Vorster (2015) asserted that C2005 was introduced to address educational imbalances and to produce the more skilled workforce needed by the new government. Unlike its predecessor (NATED 550), which prescribed content, C2005 did not concentrate on the content to be taught at all. This curriculum favoured schools with highly qualified teachers and equipped with other educational resources because the teachers could easily perform the roles expected under OBE. Most of the previously disadvantaged schools lacked both human and infrastructural resources at the inception of C2005.

Cobbinah and Bayaga (2017) argued that any curriculum conveys the policies that enable the country to measure the quality of its education against international standards and it influences classroom practices. Classroom practices are dependent on the teacher’s learning experiences. Adler and Davis (2006) noted that mathematics teachers from previously disadvantaged communities in South Africa fail to unpack mathematical ideas. They argued that this is a direct consequence of their training which emphasized the compression of mathematical ideas.

In the old curriculum (NATED 550), teachers were the holders of mathematical knowledge which was to be transmitted to passive students. Mathematics was taught in isolation of the learners’ experiences and everyday life. There was no room for nurturing learners’ informal problem-solving strategies. The old curriculum, and mathematics, were aimed at selecting a few students for university education and white-collar jobs. In post-apartheid, teachers are expected to be facilitators in the learning process, by creating conducive learning environments and designing challenging problems, relevant to the real world, and to launch mathematical ideas. The post-apartheid curriculum intended to equip learners with the knowledge, skills and values necessary for full participation in society (Froneman et al., 2015).

There have been numerous modifications to the South African School curriculum since 1998, which resulted in the advent of National Curriculum Statement – Curriculum and Assessment Policy Statement (NCS – CAPS) in 2011. However, the supporting philosophical assumptions and principles, and, in particular, outcome-based education, remained. In mathematics, the change was marked by inclusion of new topics, namely data handling and financial mathematics, across all grades.

For the first time mathematics became compulsory at school. Learners in the FET band have a choice between mathematics and mathematical literacy, but must take one of these to the end of their secondary education. The new curriculum also signalled a change in assumptions with regard to the philosophical foundations that underpinned the teaching and learning of mathematics at school level. Froneman et al. (2015) asserted that in the new era mathematics is seen a product of human activities evolving from sense-making and problem-solving activities. NCS – CAPS, unlike C2005, prescribes the content to be learnt per grade and specifies four cognitive levels (viz. knowledge, routine procedures, complex procedure and problem solving), which should be considered when assessing learners.

Despite these changes in policy, Gaigher et al. (2006) have argued that most black learners still attend schools that are at a disadvantage due to shortage of qualified teachers, large classes and poor facilities. These learners as Loji (2012) pointed out enter initial teacher education (ITE) programmes deficient in problem-solving skills. It is questionable whether the situation would have changed when these students exit ITE programmes.

2.4 Educational Inequalities

Almost thirty years into the new dispensation, South Africa has not completely rid itself of its racially divided apartheid schooling system. South Africa does not only have the most unequal society in the world but also the most unequal schooling system (Maree et al., 2006). Educational inequalities are highlighted by the wide score distributions in TIMSS 2003 to TIMSS 2015 (Reddy et al., 2016). Expanding upon this point further, (Reddy et al., 2013, p. 1) stated that:

> *South Africa has two types of public schools, which continue to be affected by socio-economic factors inherited from the Apartheid period: Public schools* (mainly attended by African learners), *located in areas with the lowest economic status where the majority of Africans live; and independent/multi-racial schools constituted by formerly white schools, Indian schools, and "Colored" schools*.

Berger, Bowie, and Nyaumwe (2010) added that two examination boards, DOE and IEB, exist in South Africa. The majority of schools in South Africa are public institutions and they mostly write the DOE examinations, while private schools write IEB examinations. The combined educational performance of South African schools on international assessments, such as TIMSS 2011 and TIMSS 2015, is way below the international mean of 500 points (Arends et al., Feza, 2014b). However, when disaggregated, the data show that independent schools' performance is very close to the international mean (see table 1).

Table 1: SA Schools' Performance in TIMSS 2011

School type	Math	Science
Independent	474 (17.1)	479 (19.0)
Public	348 (2.5)	327 (3.7)
South Africa	352 (2.5)	332 (3.7)
Quintile 1	316 (5.7)	279 (7.9)
Quintile 2	318 (3.6)	285 (6.4)
Quintile 3	336 (1.0)	314 (5.9)
Quintile 4	360 (5.6)	348 (7.6)
Quintile 5	438 (9.7)	445 (12.6)

Source: Adapted from Reddy et al. (2013).

Public schools are classified into quintiles based on the socio-economic status of the society in which the school is found. Quintile 1 is the poorest, while quintile 5 is the richest. The poorest schools are the worst performers in mathematics. It is not the intention of this study to describe or critique the educational performance of South Africa but this table is included to show that the institution in which this study takes place mainly draws its student population from quintile 1 to 3 as a result of its location and to highlight the environment in which these teachers are likely to be found after completion of their studies. Hence, it is essential that their learning experiences are established so as to make recommendations on how to strengthen ITE programmes and make suggestion on ways in which these teachers could be supported through in-service interventions to improve learning in the underperforming quintiles.

The performance of independent schools reflects the maxim of "once privileged always privileged". Adler and Davis (2006) asserted that teachers in these schools are highly qualified since during apartheid only white students were allowed to do full undergraduate training and a Post Graduate Certificate in Education (PGCE) to qualify for secondary school teaching, while black students were confined to under resourced colleges of education to qualify as teachers. Teacher education has undergone numerous reforms since the advent of democracy, the latest being the alignment of all ITE programmes as dictated by Minimum Requirements for Teacher Education Qualification framework (MRTEQ) but the results are yet to be realized in learner performance.

This study sought neither to understand the socio-economic status of prospective teachers nor to evaluate learner performance but to explore PSMTs non-routine mathematical problem-solving learning experiences in financial mathematics, with the hope that the findings would be used to influence curriculum changes and stimulate debates on the extent to which this institution supports national goals on problem solving.

The majority of PSMTs participating in this study come from public schools in quintiles 1–3, which are known to be underperforming, so knowing their problem-solving experiences could assist lecturers in the faculties of education in revolutionizing their practices so as to further the goals of the fourth industrial revolution and the goals of the South African National Development Plan 2030.

In the sections that follow, this researcher has drawn on mathematics curriculum policy to illustrate the expectations of the curriculum for the envisaged teacher and learner. The philosophy underpinning the school curriculum for mathematics is also discussed.

2.5 Mathematics

2.5.1 Mathematics' definition and its implication for teaching and learning mathematics

According to the Department of Basic Education (2011b, p. 8):

> *Mathematics is a **language** that makes use of symbols and notations to describe … relationships. It is a **human activity** that involves **observing**, **representing** and **investigating patterns** … It helps to **develop mental processes** that enhance logical and critical thinking; accuracy and **problem solving** that will contribute in decision making.* (bold added for emphasis)

The DBE seems to take a tripartite stand on how school mathematics should be viewed. Firstly, mathematics is studied as a discipline which *develops mental process*, i.e. mathematical gaze/cognitive value, and secondly, mathematics is studied due to its utilitarian/applicability benefit for mankind (mathematics is used to solve problems encountered by mankind) that is livelihood gaze. Thirdly, learning mathematics should enculturate learning mathematicians into human activities of identifying and solving problems by engaging in the practices of mathematicians. Addressing these seemingly different positions would require intensive teacher training. These implied positions on mathematics teaching and learning places special demands on the envisaged teachers' philosophy of mathematics, subject matter knowledge and subject pedagogical knowledge (Cotton, 2013).

South Africa seems to have aligned itself with international trends in the teaching and learning of mathematics. Its definition of mathematics can be traced back to the Freudenthal Institute of Mathematics in the Netherlands where the conception of mathematics as a human activity was started in the seventies (Gravemeijer & Terwel, 2000).

Being a human activity, the teaching and learning of mathematics implies mathematics:

- Must be relevant to society.
- Must enhance opportunities for guided mathematical re-invention.
- Should foster both horizontal and vertical mathematisation (Sasman, 2011).

The purpose of teaching mathematics is to gradually socialize the learner into the mathematician's way of working, as opposed to teaching the results of the mathematician's activity (Gravemeijer & Doorman, 1999; Gravemeijer & Terwel, 2000). For example, in most South African schools, learners are taught how to use a quadratic formula to solve quadratic equation problems. This approach to teaching mathematics is not only in contradiction with the definition of mathematics but it denies learners an opportunity to re-invent their mathematical knowledge. Learning mathematicians are often left wondering why the formula works and when it is appropriate to use it.

Gravemeijer and Terwel (2000) argued that such an approach is an anti-didactical inversion, as the product of others' mathematisation is used as a starting point.

The traditional approach of teaching mathematics also fails to address the assumptions underpinning the South African mathematics curriculum as it does not allow learning mathematicians to learn the processes through which mathematics is developed. The Department of Basic Education (2011b, p. 9) in its definition of mathematics says "…mathematics is developed and contested over time through both language and symbols by social interaction and is thus open to change".

For this to happen, learning mathematicians must be afforded opportunities to acquire skills and knowledge by responding to challenging and meaningful problems which extend their current level of thinking. The traditional way of teaching mathematics (emphasizing procedures without understanding, i.e. rules without reason) has led to what is locally and international called 'mathsphobia' in most learning mathematicians and the entire population (Department of Basic Education, 2011b). Teachers who use traditional approaches prematurely formalize mathematics into the abstract at the expense of developing conceptual understanding, building process skills and mathematical language skills in their students. It requires relational understanding before procedural automacy.

Teachers' learning experiences are a prerequisite for preparing learners who are mathematically competent. The question that arises from this definition for this study is: What learning experiences should pre-service mathematics teachers (PSMTs) experience prior to exiting ITE programmes so that they can teach mathematics as a human activity?

The sections that follow discuss the forms of mathematical knowledge which seem to be advocated by the South African school curriculum.

2.5.2 Mathematical knowledge within the school curriculum

As discussed in the definition of mathematics in section 2.5.1, school mathematical knowledge entails linguistic, disciplinary and strategic knowledge forms (these are elaborated in the sections that follow). The curriculum seems to expect that mathematically proficient teachers are conversant with these forms of knowledge and that they would be able to teach with these in mind. This view is supported by Cotton (2013), who argued that the envisaged 'good' teacher should possess a thorough understanding of mathematical subject matter knowledge, mathematics curriculum knowledge and mathematics pedagogical content knowledge. Hence, exploring learning experiences of pre-service mathematics teachers (PSMTs) holistically demands that one examines what teachers should know and be able to do within these knowledge domains.

2.5.3 Linguistic/social mathematical knowledge

Mathematical knowledge is socially constructed. In support, Ernest (1991), cited in Barnes and Venter (2008, p. 5), posited that:

> *1. The basis of mathematical knowledge is linguistic knowledge, conventions and rules, and language is a social construction. 2. Interpersonal social processes are required to turn an individual's subjective mathematical knowledge, after publication, into objective mathematical knowledge. 3. Objectivity itself will be understood to be social*.

Social mathematical knowledge consists of the mathematics register[6] and it enables one to engage in mathematical discourses. Learning mathematics as a language means knowing how to use mathematical language to communicate in various situations by using language (words, symbols, notations and conventions) and representations (visuals, graphs and algebra) that are concise, unambiguous and appropriate to the context. As pre-service mathematics teachers (PSMTs) solve non-routine mathematical problems in financial mathematics, they converse within and across the problem, sharing their understanding of the problem, problem-solving plan and problem-solving strategies by using mathematical language and representation. Learning and teaching mathematics may be challenging and problematic, with many linguistic issues. For example, the meaning of 3 in y^3 or $3x$ or $3!$ is not the same. The varying meaning associated with the same concept compounds the difficulties faced by problem solvers. Language plays a key role in learning and solving mathematical problems. It relays concepts and

[6] This researcher has used the term 'register' as defined by Herbel-Eisenmann, Johnson, Otten, Cirillo, and Steele (2015).

procedures. Mathematical problems in financial mathematics (FM) often use technical vocabulary (such as semi-annual, discount, and effective interest rate). Language may also help PSMTs to solve non-routine problems verbally through logical reasoning, definitions and inferring. Language can help learning mathematicians to deepen their understanding of financial mathematics' concepts but it can also cause misconceptions. The biggest challenge which confronts problem solvers in FM is that they need to understand both the technical language and the mathematics content. If teachers are expected to teach mathematics as a language, an important question is which linguistic learning experiences should PSMTs demonstrate adequately as they exit ITE?

Mathematical language is used to construct, express and communicate mathematical ideas. Language[7] is critical to successful non-routine mathematical problem solving in that problem solvers have to translate the text in the problem to mathematical symbolic language or represent the information visually as part of solving the problem. Any misinterpretation of the problem may interfere with the problem-solving process. Unlike in most mathematical topics, where problems are presented in well refined symbolic form, such as solve for x in $x^2 + 2x + 1 = 0.$, mathematical problems in financial mathematics are frequently presented in words and rarely in graphical or algebraic form. Hence, success in solving problems in financial mathematics depends on one's ability to fluently communicate mathematical understanding using the various modes (words, symbols, tables and graphs). Maree et al. (2006) argued that South African learning mathematicians who participated in international studies struggle with problems requiring them to transit from verbal language in the problem to symbolic mathematical language that is often used to solve contextualized problems.

To mathematically solve problems presented in words, the problem solver has to form an accurate (mental) representation of the quantitative elements, the relationships amongst them and then formulate a mathematical model which would be used to solve the problem. Bernardo (2002) warned that linguistic demands in a mathematical problem

[7] In this study, the term 'language' is used to refer to language of mathematical not merely natural language.

interfere with problem representation, resulting in errors in the solution. In support, the Department of Basic Education (2014, p. 116) pointed out that:

> *As language presents problems in the correct interpretation of finance questions, teachers must at all times focus on using the correct language…teachers need to expose learners to all terminology applicable to financial mathematics while teaching and in setting assessment tasks.*

Language has more adverse effects on the problem solvers' ability if the problem is presented in a language different from the problem solvers first language (Bernardo, 2002). All the participants in this study have learned and would be teaching mathematics in English which is not their mother tongue.

Trinick, Meaney, and Fairhall (2014) asserted that teaching mathematics in a language in which one is not fluent has implications for pedagogy, learner performance and attitude to mathematics. After analysing the practice of one primary school teacher in a multilingual classroom, Setati (2005) concluded that whenever the teaching was conducted in the second language, the emphasis was on mathematical procedures but whenever the teacher switched to vernacular the emphasis was on developing conceptual understanding of mathematical concepts. Based on these researchers' views, it can be concluded that for one to teach for relational understanding/meaningfully learning (Skemp, 1986), one needs to be fluent in the language of teaching and learning (LoLT), in addition to being proficient in mathematics and mathematical language.

Linguistic fluency is critical in this study, since PSMTs need to interpret and represent the problem in an equivalent mathematical format, which takes into account all the problem conditions necessary for solving a problem. Constructing a mathematical representation of a problem and consequently developing a problem-solving strategy is wholly dependent on the problem solver's ability to understand the language in the problem. In this study, PSMTs' linguistic challenges are minimized by allowing PSMTs to work together in solving mathematical problems.

In contrast, Englard (2010) noted that young learning mathematicians are able to construct strategies for solving problems which takes into consideration actions and relationships, though it is not clear whether the problems were in their home language or second language. However, Englard (2010) noted that as their mathematical ability develops, they start looking for superficial cues such as key words or operations. Learning mathematicians' use of cues in problem solving was also noted by the Department of Basic Education (2012) who warned that "candidates…tend to associate a formula with a certain form of phrasing in the question… once they saw the word 'invest' in the question, they thought it meant the use of future value formula…".

Olivier (1989) added that during mathematical problem-solving processes, some item(s) of the information (e.g. words, form of equation, function, number of terms, etc.) acted as a cue in triggering retrieval of an appropriate conceptual structure (schema). He argued that the cues, problem content and the appropriateness of the schema are critical to successful problem solving.

Mathematical language consists of symbols, notations and conventions which mathematically proficient problem solvers use (Kilpatrick et al., 2001; Talasi, 2007). Mathematical language differs from ordinary language in that it attaches specialized meaning to everyday words and when written it is mainly symbolic. When learning mathematicians engage in non-routine mathematical problem solving, they move between ordinary and mathematical language in search of a solution. Learning mathematicians with less command of the language used in the framing of the problem maybe be alienated during problem solving. In support, Maree et al. (2006) advanced that most South African learners who participated in TIMMS struggled to solve problems involving language. They further pointed out that apart from learners' weak mathematical knowledge, learners had problems in communicating their answers in the language of the test (English). Problem solvers' mathematical language may affect problem interpretation and conceptualization.

In solving non-routine mathematical problems in financial mathematics, the written text (usually in everyday language) provides the contextual information. The problem solver synthesizes the text, and uses mathematical symbolism to describe relationships between variables by using diagrams (such as a timelines) to connect the real world to mathematical processes (Schleppegrell, 2007). Therefore, it is of paramount importance that the problem solver is linguistically fluent (in both the everyday language and the symbolic mathematical language) to meaningfully solve the problem. This would help the problem solver in using the written language, mathematical symbolism, visual representation and oral language to construct and share knowledge with others as they interact and discuss the problem. During problem solving, the problem undergoes a transformation. The transformation of a problem demands that the problem solver uses various modes of communication flexibly (oral, algebraic, visual, etc.) while retaining the same meaning. Thus, from a linguistic point of view, problem solving is not just a question of manipulating symbols on a piece of paper but a process of constructing meaning individually or in collaboration with others.

This study also endeavours to determine the actions taken by PSMTs when they confront linguistic challenges during problem solving. The actions were classified as linguistic learning experiences, since PSMTs constructed meaning of words by either defining terms, such as ‘semiannual’, or explained the meaning of phrases, such ‘a debt is retired’. These actions form part of PSMTs’ problem-solving experiences as they result in the expansion of their current knowledge or the construction of new knowledge. Olivier (1989) expounded that mathematical knowledge can be thought of as being logico-mathematical (sometimes referred to as conceptual knowledge), physical knowledge and social knowledge. He further posited that social knowledge is learnt through social interactions. Hence, PSMTs’ verbal communication is critical in meaning construction and ultimately in resolving the problem. In section 2.5.4, subject matter knowledge is elaborated upon further by looking at disciplinary knowledge.

2.5.4 Disciplinary mathematical knowledge

Disciplinary knowledge consists of logico-mathematical knowledge (concepts and procedures) (Olivier, 1989), skills and mathematical practices/processes (such as making and justifying conjectures, representing, observing etc.), which students must acquire and be able to apply in various mathematical contexts. Kilpatrick et al. (2001) added that mathematical knowledge is made up of algebraic activities (section 3.2), while Usiskin (1988) viewed disciplinary mathematical knowledge as being made up of arithmetic and algebra. Engaging in mathematical practices such as investigating patterns, formulating, testing and generalizing conjectures helps learning mathematicians to develop lateral thinking skills. In support, the Department of Basic Education (2011a, p. 9) argued that "a suitable range of mathematical process skills and knowledge enables an appreciation of the discipline itself and also ensures access to an extended study of the mathematical sciences and a variety of career paths".

Unlike social knowledge, disciplinary knowledge demands one to construct it. Teachers can facilitate its construction by arranging situations to be mathematised by learners (Barnes & Venter, 2008). These situations should be carefully selected and sequenced so as to enable conceptual development. Makonye (2014) added that disciplinary knowledge is best acquired by learners when it builds from learners' current informal strategies and everyday experiences which are gradually formalized into abstract mathematics.

Mathematical procedures and skills are not immune to context. A knowledge of algorithms and skills, such rounding off, may prove to be inadequate for teaching mathematics when the problem context is considered. For example, in a problem such as:

> *17 people are trapped on a mountain and need to be rescued by helicopter. The helicopter can take a maximum of 4 passengers at a time, in addition to the pilot. How many trips will the helicopter need to make?*
>
> And
>
> *17 ÷ 4 = ? Round off your nearest whole number.*
>
> (Barnes & Venter, 2008, p. 4)

The same algorithm is used to arrive at the different answers (5 trips and 4 respectively). This may be confusing to novice problem solvers because the first problem deviates from the classroom norm of rounding, i.e. if the digit on the immediate right side of the required digit is less than 5, round down, and round up for digits from 5 to 9. The above problems highlight the difference between contextualized mathematics and esoteric (decontextualized) mathematics.

Traditionally, at school, mathematics procedures are taught in the mathematical realm with no connection to everyday experiences of learning mathematicians. The decontextualization of mathematical problems and solutions by learners result in meaningless answers. However, solving problems in financial mathematics requires that problem solvers interpret the answer in relation to the context.

Da Silva et al. (2009) found that learners solving problems in financial mathematics made three types of errors, namely mathematical, representational and financial contextual errors. They defined **mathematical errors** as the error which relates to the mathematics in a calculation or strategy, irrespective of context. **Representational error** refers to the way learners communicate their ideas and calculations but such errors still lead to correct numerical answers. A **financial context error** is strongly linked to the financial context, and it relates to the way mathematical skills, such as rounding off, are applied in various mathematical contexts. The question is what constitutes the repertoire of PSMTs' learning experiences that would enable them to deal with mathematical problems that place different cognitive demands on learners as illustrated by the problems stated above, while at the same time minimizing learner errors during problem solving to enhance their performance.

2.5.5 Strategic mathematical knowledge

Strategic mathematical knowledge consists of problem-solving heuristics and problem-solving strategies (Krulik & Rudnick, 1982; Polya, 2014; Posamentier & Krulik, 2008). Kilpatrick et al. (2001) added that strategic knowledge is the knowledge needed to formulate, represent and solve mathematical problems using logical thought, reflection, explanation and justification. This form of knowledge is often not taught in schools. However, strategic knowledge is the most commonly used knowledge, post schooling years. After studying 105 Grade 10 mathematical literacy learners solving problems in financial mathematics, Khalo and Bayaga (2014) concluded that learners' errors could be minimized by teaching learners to use Polya's problem-solving heuristic.

Financial mathematics as a topic within school mathematics is largely linguistic as it deals with loans and investments which are represented as text. Problems in this topic often require the problem solver to be conversant with the technical language (terminology) used in formulating responses. Unlike solving problems in other areas of mathematics, solving problems in financial mathematics demands that the problem solver transits from the world of words (text) into the mathematical world of symbols and/or visual representations. This transition demands that PSMTs are able to decompress and compress (Adler, 2005) mathematics during problem solving. Decompressing and compressing mathematics (Adler, 2005) places teaching expectations on envisaged teachers, which draw on mathematical philosophies. The following section provides a brief overview of the philosophy of the mathematics teacher envisaged by the Department of Basic Education.

2.6 Philosophy of the Envisaged Mathematics Teacher

The philosophical views on the nature of mathematics lie in a continuum. Makonye (2013) posited that extreme positions on the nature of mathematics are absolutism and fallibilism. He further added that absolutist philosophies of mathematics include logicism, formalism and Platonism.

Absolutism views mathematical knowledge as absolute, immutable and existing external to human mind, and hence mathematical knowledge is certain, objective and true (Ernest, 1991; Handal, 2003; Makonye, 2013). From this perspective, mathematics is an inert body of knowledge which should be transmitted to novices (Cotton, 2013). Problem solving in this sense is discovering mathematical truths which exist external to the human mind (Makonye, 2013), while teaching mathematics' progresses through telling novices about the existing facts. Consequently, doing mathematics means mimicking the teacher's methods and repeating them on several similar problems.

In contrast, fallibilists hold a view that mathematics is a product of human creation, solely intended to solve the practical and theoretical problems encountered on daily basis (Makonye, 2013), and hence it is open to revision, based on new developments (Handal, 2003). Fallibilism is a constructivist philosophy arguing that people construct knowledge by forming relationships amongst mathematical ideas. Thus, all mathematical knowledge is an outcome of human endeavours and it cannot exist independent of them, as absolutists argue. Fallibilism recognizes the socio-cultural and historical development of mathematics as paramount in explaining the nature of mathematics. From a fallibilistic position, mathematical knowledge is dynamic and learning mathematics means engaging in mathematical activities, such as observing, investigating, formulating and justifying conjectures. Learning mathematics entails solving relevant and challenging mathematical problems for which no known algorithm is readily available (at least to the problem solver). Learning mathematicians' informal problem-solving strategies are interrogated and refined. In support, Siswono, Kohar, and Hartono (2017) added that teaching mathematicians need to plan and execute classroom discussions that allow learning mathematicians to share multiple problem-solving strategies, and explore inconsistencies and congruencies in the strategies to deepen mathematical insight.

Falliblist's view mathematical problem solving as a process of solving relevant non-routine problems using informal methods. The Department of Basic Education (2011a) posited that mathematics is developed and contested over time through both language and symbols by social interaction and is thus open to change. Therefore, the DBE appears to envisage teachers who hold a fallibilistic view of mathematics so that their classrooms become platform for exploration, experimentation, and justification of mathematical ideas. Such a view seems to align with teaching mathematics through problem solving and lifelong learning.

In section 2.5.1, this researcher indicated that the DBE's orientation to mathematics teaching can be traced back to the Netherlands. The Netherlands use of a progressive philosophy in the teaching of mathematics dates back to Freudenthal (1973), who argued that mathematics teaching should start with solving realistic problems that allows for mathematical reinvention (Gravemeijer & Doorman, 1999; Gravemeijer & Terwel, 2000). It was further argued that the DBE seems to envisage teachers who adopt a fallibilist philosophy in mathematics teaching, as opposed to the absolutist philosophy, which is found to be dominant in most disadvantaged schools in South Africa (Barnes, 2004).

The adoption of a fallibilistic orientation to teaching mathematics demands highly knowledgeable teachers across schools to guarantee that the same concepts, skills, facts, values and attitudes are developed in learners. At the start of democracy in South Africa, in 1994, such teachers were not available in previously disadvantaged communities to implement C2005. Hence, the learning experiences of learners continued to be underpinned by traditional teaching, which emphasized rote learning and the absolutist philosophy. The question that arises for this study, therefore, is to what extent should the learning experiences of pre-service mathematics teachers be transformed so that they can teach mathematics as a human activity rather than a collection of rules?

2.7 Importance of Mathematics Within the South African School Curriculum and Beyond

Since the inception of C2005, mathematics teaching and learning has become compulsory for the first nine years of schooling, with learners having to choose between mathematics and mathematical literacy in the last three years of schooling. The move to include teaching mathematics to all children in South Africa is in stark contrast to the belief that there was no need to teach "mathematics to a black child if he or she could not use it in a career" (Giliomee, 2012, p. 80). In the past, during apartheid, learning mathematics was seen as only necessary for participation in specific, racially-defined careers and government affairs. Feza (2014b) added that mathematics was used to exclude black South Africans from employment and furthering their studies in STEM (Science, Technology, Engineering and Mathematics), as well as in accounting, business, economic and management fields.

The importance and exclusive nature of education, in general, and mathematics, in particular, cannot be overemphasized in societal advancement. Similar sentiments are shared by Giliomee (2012, pp. 72-73), who cited Hendrick Verwoed's 1953 parliamentary speech:

> *Education should have its roots entirely in the Native areas and in the Native environment and the Native community… The Bantu must be guided to serve his own community in all respects. There is no place for him in the European community above the level of certain forms of labour. Within his own community, however, all doors are open.*

This shows how indigenous people were kept on the periphery of education and of learning mathematics. Hendrick Verwoerd's comments in 1953 about teaching mathematics to black children and the subsequent removal of qualified white teachers from missionary schools impacted negatively on black communities, as mathematics was no longer taught as a compulsory subject, resulting in the low provision of high quality mathematics teachers (Giliomee, 2009). The shortage of teachers made mathematics learning inaccessible and where it happened, it was dominated by rote learning. This fostered the belief that mathematics was difficult and that only a few could learn it.

While there has been a change in the political landscape of South Africa, since the end of apartheid in 1994, mathematics has continued to play a gate-keeping role for most black children, restricting them from entering scientific, economic and financially-based careers and from furthering their studies in these fields. This necessitates that faculties of education take stock of the learning experiences of current PSMTs so as to identify their level of readiness and areas that need to be revamped in ITE programmes to ensure that mathematics is taught as intended by curriculum developers, and thus enabling the realization of the DBE curriculum aims.

Under the new dispensation, mathematics is seen as a tool necessary for partaking in adult life activities, such as employment, citizenship, and community living, as well as the participation in both local and global ever-changing social, economic and technological affairs. The Department of Basic Education (2011b, p. 4) added that teaching mathematics to all learners is aimed at:

- *equipping learners, irrespective of their socio-economic background, race, gender, physical ability or intellectual ability, with knowledge, skills and values necessary for self-fulfillment, and meaningful participation in society as citizen of a free country…*
- *providing access to higher education;*
- *facilitate transition of learners from education institutions to workplace.*

In most of the previously disadvantaged communities in South Africa, mathematics is still seen as an elite subject. A proficiency in mathematics is still associated with being incredibly clever, ignoring the fact that success in mathematics is also dependent on how one comes to know mathematics.

The Department of Basic Education (2011b) added that mathematics has both intellectual and social benefits, since it enhances logical and critical thinking skills and it improves the learning mathematician’s accuracy and problem-solving skills. Mathematics provides the tools and language for generalizing observed patterns. It functions as a language in the natural sciences, economic and management sciences and information technologies. In addition to these, mathematics is required for most careers, and for pursuing further studies in many fields, as well as for everyday living. In support, Alausa (2000) asserted that the choice for a science subject at a certain level is related to one's mathematical ability and that mathematics has contributed to banking, finance and insurance, in the economic and management sciences. In addition, mathematics equips learning mathematicians with the necessary tools to collect, organize and analyse quantitative data, evaluate and critique conclusions, understand and describe the world, and to deepen their understanding, while enhancing their ability to solve real-world problems (Department of Basic Education, 2011a; Sasman, 2011). Therefore, the teaching and learning of mathematics is of paramount importance for any nation that wants to advance and compete in the global market. In agreement, Alausa (2000) posited that mathematics is recognized by both developed and developing countries as a critical ingredient for socio-economic development.

Pivotal to mathematics learning are initial teacher education (ITE) programmes which provide the Department of Basic Education (DBE), and the schools, in particular, with mathematics teachers for the implementation of the curriculum. It is thus essential for institutions offering an ITE programme to ensure that their programmes are up-to-date and relevant to both the local and international requirements and expectations for best practices in the teaching of mathematics.

Despite this, Sasman (2011) and Bansilal et al. (2010) pointed out that the mathematical skills and abilities of most South African school leavers are a cause for concern. The same sentiments are shared by Spaull and Kotze (2015, p. 8), who argued that "the state of mathematics education in south Africa is dire as indicated by poor learner performances in various international studies". They further indicated that insufficient teacher knowledge is of great concern as teachers' performance on items similar to those written by their learners was also low. Despite a wave of curriculum modifications both at the policy level and at school level, learners' performance is yet to make an impact on international assessment. Learners' lack of problem-solving skills is seen as the main contributor to underperformance (Nieuwoudt, 2015a).

Critical to any curriculum innovation is the teacher factor. In support, Krulik and Rudnick (1982) argued that to enhance learners' problem-solving skills, teachers have to be extensively trained in this crucial skill. It is the responsibility of teachers to create a conducive learning environment and monitor learning. To achieve this, teachers must develop, adopt or adapt mathematical problems which challenge and motivate learners to use their current mathematical knowledge in different ways in search for a solution.

Krulik and Rudnick (1982) expounded that an ITE programme should take prospective teachers through the same journey they would be expect to take their learners. Lack of problem-solving skills amongst learners and, to some extent, their teachers, has prompted government, researchers, academics and the Department of Education to embark on identifying ways of improving the teaching and learning of mathematics in schools. This researcher therefore wished to know what learning experiences do pre-service mathematics teachers need to go through in the context of their solving of non-routine mathematical problems in financial mathematics.

The South African mathematics school curriculum is divided into five content areas: Number and number relationships; functions and algebra; space, shape and measurement and data handling (Department of Basic Education, 2011b). This study explored pre-service mathematics teachers learning experiences in solving non-routine problems in financial mathematics. Financial mathematics is placed under number and number relationships in the General Education and Training band (GET) and under functions and algebra in the Further Education and Training band (FET). Hence pre-service mathematics teachers' learning experiences were explored across these topics.

2.8 Financial Mathematics Within the South African School Curriculum

Globally mathematics is regarded as the foundation of all scientific and economic developments. It is the language of business and it is found in all spheres of human endeavours. It is now compulsory that citizens are mathematically and financially literate since they are consumers of financial products which are mainly communicated mathematically. However, mathematical literacy[8], in general, and literacy in financial mathematics, in particular, cannot be achieved without having knowledgeable mathematics' teachers in those areas.

The ability to solve non-routine financial mathematics' problems at school is an important tool in managing one's finances, towards attaining financial security and responsibility, in promoting financial literacy skills and increasing independence (Root et al., 2017). In this regard, quality instruction in solving financial mathematics' problems, such as related to saving, investing, lending and borrowing, is the first step to personal finance problem solving in current (i.e. school) and future environments.

According to Lusardi and Mitchel (2011), financial illiteracy knows no boundaries, being widespread across all economies, gender and age groups. They argued that people who are financially literate are more likely to make more informed decisions about household finances, as well as planning for their retirement. The inclusion of financial mathematics within the South African school curriculum is critical in that it introduces young adults into the lived experiences of adult life regarding saving, investing, borrowing and spending. A knowledge of mathematics and financial mathematics is useful when one engages in these activities since one needs to make informed decisions about the different loan or investment options available.

In a country where unemployment is continually increasing and about 31% of the population is living off some form of government grant, financial understanding is of paramount importance (Engelbrecht, 2014). However, financial literacy cannot be attained without relational understanding and fluency in mathematical procedures (Kilpatrick et al., 2001).

Pournara (2015a) stated that financial procedures learnt at school are not always used by the major banks in South Africa. The case in point is the use of effective interest rates at school to the exclusion of an average interest rate when dealing with investments.

[8] Mathematical Literacy. This researcher notes that in South Africa the term mathematical literacy refers to a school subject studied by learners who intend to follow careers which do not require 'pure' mathematics, whereas in other countries, it refers to the competence which citizens should demonstrate. This researcher uses this term here in the latter sense.

Pournara (2015a) argued that the latter is used by banks when advertising fixed deposits to lure customers into investing as it shows higher percentages. Unsuspecting or financially illiterate costumers would think that in table 2, interest on expiry would give a higher yield on their investment than the other options due to a higher interest rate. However, calculations show a different outcome. As a mathematics teacher, one wants to know the kind of learning experiences particularly problem-solving strategies which PSMTs need to know and be able to use in practice when teaching financial mathematics.

Table 2: Nedbank Investment Options

Nedbank Green Savings Bonds
18, 24, 36 or 60 month deposits, for clients less than 60 years of age.

Balance	Period of Investment	Interest Monthly	Interest Half-yearly	Interest on Expiry
R1 000 or more	18 months	6.90%	7.00%	7.24%
	24 months	7.14%	7.25%	7.65%
	36 months	7.39%	7.50%	8.24%
	60 months	8.11%	8.25%	9.96%

Source: Adapted from Pournara (2015).

Unlike in most countries, which treat financial literacy outside mathematics, in South Africa, financial literacy is part of both mathematics and mathematical literacy[9], under the topic named financial mathematics. It may therefore be concluded that financial mathematics has a dual purpose within the South African school curriculum, namely to develop a disciplinary knowledge base needed for further studies in careers such as actuarial science and to eradicate financial illiteracy amongst citizens. What is not clear is what kind of learning experiences would the envisaged teacher need to be exposed to during training to facilitate learning activities that address two seemingly divergent curriculum goals. Pournara (2011) added that financial mathematics within the school curriculum provides a context in which mathematics is applied in the real world and also serves to introduce learners to what happens in the banks.

[9] In South Africa, the term mathematical literacy refers to both the school subject Mathematical Literacy (ML) and the competency in Mathematics (mathematical literacy (ml)).

According to the Department of Basic Education (2011b), school mathematics is divided into five main content areas: Number and number relationships; functions and algebra; space, shape and measurement and data handling. Financial mathematics is not a 'stand-alone' content area but it is incorporated in number, and number relationships (Department of Basic Education, 2011b). Learners are expected as early as Grade 7 to solve problems that involve financial contexts such as loans, simple interest, and budgets. while Grade 12 learners are expected to use geometric series to solve problems involving annuities and critically analyse different loan options (Department of Basic Education, 2011b).

Unlike most mathematics topics which have their own content, financial mathematics contextualizes and applies content and skills from other topics to the context of banking. As an applied area of mathematics, the teaching of financial mathematics requires highly-skilled teachers for its successful incorporation into the school curriculum (Pournara, 2013b). However, financial mathematics is not always understood by mathematics' teachers, resulting in learning mathematicians leaving school ill-prepared for the consumption of financial products during adulthood (Bhandari, 1997; Makonye, 2017; Pournara, 2013b). For example, many learners think that banks use formulae such as $F_v = P_v(1+i)^n$ or $F_v = R\frac{[(1+i)^n - 1]}{i}$, which they struggle to understand and use at school. Learning mathematicians may think that investing R50 at a rate of 6% p.a. compounded monthly means one would get equal interest on a month to month basis. Taking this point further, Parramore (2011) argued that for most learning mathematicians, and teaching mathematicians, solving problems that deal with loan repayments is always an insurmountable challenge. In most classrooms, learning financial mathematics has been reduced to the memorization of senseless mathematical formulae that work by chance. The downfall of such teaching is that it stalls relational understanding, even though it may help learners attain good marks in tests and examinations (Makonye (2017).

Pournara (2011) posited that financial mathematics would be learned better if learning mathematicians were taught to understand the iterative processes that are undertaken by the banks and if mathematical formulae were used to predict the accumulated amount. Pournara (2011) argued that in order for FM to be taught with understanding, teaching mathematicians need to understand how the world of banking operates, the terminology and the conventions used in financial contexts. For example, rand-based markets use 365 days to allocate interest, while the US-based markets use 360 days. These conventions are used irrespective of whether a year is a leap year or not. Clarifying banking practices which are not emphasized at school but are dominant in banking, Parramore (2011, p. 211) added that:

- *Interest is charged or credited on a daily basis, usually on the balance of the account at the end/beginning of the day.*
- *The balance of the account is adjusted for withdrawals and deposits, but not for interest, at the end of each day.*
- *The interest charges or credits are recorded separately. The accumulated interest is consolidated with the account at specified intervals.*

Investigating pre-service mathematics teachers' learning experiences in financial mathematics demands that the two content areas, i.e. number and number relationships and functions and algebra, be considered constituting the basis for the teachers' financial mathematics knowledge. When dealing with number and number relationships at school, learners as early as Grade 7 study patterns with the intention to generalize them or find a model that could be used for prediction. The relationships could be linear or exponential. In learning or solving financial problems, linear functions (with the form $y = mx + c$) are used in the context of simple interest and straight-line depreciation (with the form $F_v = pi \pm p$) of assets, while exponential relationships ($T_n = ar^n$) are found in compound interest and reducing-balance depreciation ($F_v = p(1 \pm i)^n$) of assets (Pournara, 2013a).

However, when financial mathematics is presented in most school textbooks, the starting point is often a formula or a definition of the terms used in the financial contexts, with no direct link to other mathematical concepts, such as arithmetic progression, geometric progression, or percentages, just to name a few. The invisibility of the link to other mathematical concepts or learning mathematicians' everyday experiences encourages the rote learning of financial mathematics.

Financial mathematics is allocated about 10% of the total marks in the National Senior Certificate (NSC) exit examinations for mathematics. However, National Senior Certificate Diagnostic Reports for the period 2011 – 2017 show that learners experience difficulties in responding to finance-based questions (Department of Basic Education, 2011c, 2012, 2013, 2014, 2015, 2016, 2017). The difficulties stem from difficulties with language, mathematics, technology and lack of practical knowledge of how loan repayments are handled on a daily basis. Language seems to interfere with the understanding, interpretation and conceptualisation of problems. As a result, students' abilities to set up a mathematical model or choose the correct formulae from a formulae sheet is negatively affected (Department of Basic Education, 2016).

Sasman (2011) noted that rushing students into vertical mathematization (abstract thinking), such as using formulae and standard algorithms, leaves learning mathematicians ill-prepared for solving non-routine problems. Rushing learning mathematicians into vertical mathematization further gives the impression that doing/learning mathematics is all about choosing formulae which work by chance, instead of developing learning mathematicians' problem-solving abilities. In the cases where the mathematical model employed is correct, students are found wanting in algebraic manipulations, basic mathematical skills or in using technology to arrive at the answer (Department of Basic Education, 2015).

Sasman (2011) argued that mathematically proficient learning mathematicians should be able to communicate their mathematical understanding fluently using words, tables, graphs and symbols. Da Silva et al. (2009) added that a flexible understanding of financial mathematical concepts could be developed through the use of multiple representations, such as numerical, algebraic, verbal and graphical representations. The use of multiple representations in learning allows for different choices of problem-solving strategies. For example, when solving for an unknown interest rate in situations involving annuities, learning mathematicians may use numerical methods (tables, estimation, intelligent guessing and testing), verbal or graphical methods, rather than the commonly used algebraic methods. The consistent choice of a particular problem-solving strategy and or representation across problems and situations may be an indicator of the problem solver's inability to choose the most effective and efficient method and or representation (Da Silva et al., 2009). The choices and the variations in such choices of problem-solving strategies, or the representations used in solving non-routine financial mathematics problems, provide insight into the problem solvers' learning experiences. Such learning experiences therefore constitute the central aspect of this thesis.

In the institution where this study was conducted financial mathematics is a semester long module for pre-service mathematics teachers. However, it is not compulsory. According to Pournara (2013b), in most countries, financial mathematics is part of financial literacy and it requires teaching mathematicians who have relevant knowledge of mathematical concepts and the conventions of the banking world (Pournara, 2014). What one needs to understand is that financial mathematics at school does not always portray the actual practice of the banking sector, though it is a precursor for the former. For example, when the compounding period is not mentioned, at school it is taken to be annual but the banking sector assumes monthly compounding (Pournara, 2014). Hence, teaching mathematicians need to go beyond a knowledge of mathematics to adequately contribute to the eradication of the relatively low learner performance in this area of school mathematics in the Grade 12 examination and the low levels of financial literacy among South Africans.

2.9 Mathematics Teachers' Knowledge of Financial Mathematics

Financial mathematics, just like mathematics, only became compulsory and available to all learning mathematicians after the demise of apartheid. According to Laridon et al. (1997), in the old curriculum, only learning mathematicians taking mathematics on standard grade were taught simple interest, compound interest, depreciation and loan repayments in Standard 10 (now called Grade 12). Aligning themselves with most current school textbooks, Laridon et al. (1997) seemed to advocate teaching FM through the use of formulae supported by the use of calculators. However, it is important to note that most learning mathematicians who entered ITE prior to 2008 had limited or no knowledge of FM due to the different curricular offered at different levels (lower grade, standard grade and higher grade). Some of these grades excluded this critical topic.

The use of formulae in teaching mathematics in general and FM in particular is more likely to foster instrumental learning (see section 2.10) as it does not always take learning mathematicians' previous learning into account, rushes learning mathematicians into vertical mathematization (see section 3.8), and stifles their creativity in solving non-routine problems. A similar observation was made by Adler and Davis (2006), who found that mathematics' courses offered to in-service teachers upgrading their qualifications emphasized the 'compression' of mathematical ideas. i.e. abstract mathematics, instead of what they called 'unpacking', which is highly useful in teaching mathematics. Unpacking in this sense is synonymous with problem solving in that it relates to learning mathematics in its unrefined form (with a minimum or no level of abstractness). Most practicing teachers, just like most school text books, drive learning financial mathematics in isolation of the context where it is practiced. Thus, for most learning mathematicians, learning financial mathematics means substitution of numbers into a formula with no direct bearing on out of school activities. Most learning mathematicians and some teaching mathematicians[10] leave school thinking that financial institutions use these formulae on daily basis. In learning FM, a formula is used to predict the accumulated amount without going through the iterative daily calculations which are undertaken by banks (Pournara, 2013a).

2.10 Types of Understanding in Learning Mathematics

Kilpatrick et al. (2001) observed that there are two seemingly contesting views on how mathematics should be taught and how it should be learnt. One view proposes that learning mathematicians should be allowed to learn mathematics by investigating problematic situations. This is what Nieuwoudt (2015a) termed a progressive approach to mathematics teaching. From this perspective, teaching mathematicians only set up situations to be mathematized and facilitate learning from the side.

The other view considers mathematics to be universal and external to human minds. This is called a traditional approach to mathematics teaching (Nieuwoudt, 2015a). From a

[10] The term teaching mathematicians refers to teachers who specialize in teaching mathematics.

traditionalist perspective, teaching mathematicians must plan, sequence, organize and explain mathematical ideas to their learners. Learning mathematicians are expected to absorb, practice and remember what they have been told.

Kilpatrick et al. (2001) posited that both views are incomplete and fall short in explicating the complexity of mathematics, and of learning and of teaching. They argued that mathematics is both invented and it is discovered. In addition, they expounded that people learn mathematics on their own and from others, particularly their teachers. Thus, mathematics teaching should provide both constrained and open learning opportunities to produce mathematically proficient learners.

As outlined in sections 2.5.1 and 2.5.3, the Department of Education expects that learning mathematicians learn mathematics as a language and a set of activities (practices). Khalo and Bayaga (2015) added that mathematics could be learnt and understood either instrumentally or relationally. Instrumental understanding is the ability to apply mathematical rules or procedures without knowing why they work or when to apply them, for instance, when dividing two fractions, invert the second fraction and multiply. In studying decontextualised mathematics, e.g. rounding off, learning mathematicians are taught that if the digit on the right of the required digit is less than five one rounds down, otherwise one rounds up. This creates a problem when dealing with contextualized problems which require sense-making, rather than the blind application of rules (see section 3.3.4 for an example).

While mathematical calculations are precise, financial calculations depend on making contextual sense, in addition to mathematical precision. For example, responding to a question like:

> Thabo invested R700, at 6% p.a. compounded annually. How much is in the account after 18 months?

One needs to go beyond knowing or choosing a formula such as $F_v = p(1+i)^n$ or $F_v = p + inp$. One needs to understand that both formulae may be used together in the solution process. One must consider the contextual meaning/interpretation of what n represents, how the answer should be rounded off[11] and whether it is possible to get such an amount from the bank.

Barnes (2004) stated that mathematics learning and teaching in most South African schools has been dominated by instrumental learning. Pournara (2013a) added that instrumental understanding may help learners to excel in typical examination questions, without enabling learners to manage their finances or solve non-routine problems.

In contrast, relational understanding is characterised by a formation of connections (Beswick, 2005b) amongst various mathematical ideas and representations, between new and existing knowledge, between mathematics learned in school and everyday life and also being able to relate them to problem contexts. For example, when compounding is more than once a year, one has to figure out how much the effective interest rate per period is prior to performing calculations. OECD (2014) argued that the arrival of technology into classrooms affords teachers and their learners opportunities to engage in HOT skills, since technology handles basic calculations more efficiently and effectively.

2.11 Summary

This chapter has provided a setting for the study by defining the major construct of **learning experience**, which has framed this study. It has been argued that only the learning experiences relating to mathematics learning and teaching form the core of this study. The reforms of South African curriculum have been examined to show the trajectory that the country has taken to ensure that all learning mathematicians get access to quality mathematics learning from the onset.

The term **mathematics,** as used within the South African school system, has been defined. This researcher pointed out that the definition of mathematics found in the Curriculum and Assessment Policy (CAPS) seems to align with progressive approaches to teaching and learning mathematics.

[11] Currently the smallest denomination in South Africa is ten cents (10c). This has a bearing on rounding financial calculations since it is not practically possible to have cents, and hence the teacher has to make mathematically sound answers which may not be financially sound.

Mathematics is a critical subject. However, the curriculum expects teachers to teach topics such as financial mathematics, which they were not taught as learning mathematicians, and possibly not as prospective teachers either.

Various examination diagnostic reports show that learning mathematicians have problems in responding to financial mathematics ‘*questions*’. The reasons cited for this relatively low performance in this area include mathematical errors, misconceptions, linguistics errors, the inappropriateness of problem-solving strategies and learning mathematician’s inability to communicated their mathematical understanding.

Pournara (2009, 2011, 2013b, 2013c) examined the knowledge PSMTs needed to know and be able to use in their practice to teach FM. He argued that such knowledge should include a knowledge of banking practices, such as iterative calculations of interest, robust knowledge of related mathematical topics, such as functions (linear and exponential), sequences and percentage that are foundational to FM. He further argued for the inclusion of technology in the form of spreadsheets in ITE programmes.

While these South African researchers concurred that PSMTs’ learning experiences need to be revised, they seemed to address this issue from different perspectives and none of them considered the strategies used by PSMTs in FM.

CHAPTER THREE: THEORETICAL UNDERPINNING OF THE STUDY

3.1 Introduction

The aim of teaching mathematics is to assist learning mathematicians to develop and use problem-solving strategies insightfully, efficiently and flexibly across different mathematical problems and context (Torbeyns et al., 2009). The development of inflexible knowledge in learning mathematicians has been singled out as the major source of academic underperformance in assessments (Kolovou et al., 2011; Yazgan, 2016). Kilpatrick et al. (2001) coined the term mathematical proficiency to describe the kind of knowledge and competences that a mathematically literate person should demonstrate. Mathematical proficiency bridges the discord between instrumental and relational understanding or a traditional versus progressive teaching approach in mathematics learning and teaching by showing that the two complement each other through the five strands of mathematical proficiency.

However, mathematical proficiency fails to consider linguistic understanding, which seems to be essential in mathematics, particularly in financial mathematics, within the South African school curriculum. In addition, it does not take into account the knowledge of multiple problem-solving strategies and the ability to apply problem-solving strategies which are necessary for successful problem solving. Section 3.2 elaborates on mathematical proficiency, as it forms part of the analytical framework used in this study. Sections 3.3 – 3.8 provide a detailed account of other constructs which were used in exploring the PSMTs' learning experiences

3.2 Algebraic Activities

Kilpatrick et al. (2001) coined the term mathematical proficiency to describe the competences which learning mathematicians should acquire while learning mathematics. These competences are interconnected and they supplement each other. They identified these competences as conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. They further pointed out that when learning algebra, these competences are subsumed into the three distinct activities of algebra, viz. representational, transformational, and justifying and generalizing activities (see table 3). Problem solvers consciously or unconsciously draw on these activities in their search for a solution to a mathematical problem.

Table 3: Algebraic Activities

Algebraic activity	Description
Representational activities	- Refers to attempts to understand the problem, which includes simplifying the problem through explanation, drawing inference from text, diagrams and formulas: - Requires translation of verbal information into symbolic expressions and equations. - Requires conceptual understanding of mathematical concepts, operation and relations expressed in the verbal information. - Requires making assumptions about the context and variables. - Requires strategic competence to formulate and represent information with algebraic equations and expressions.
Transformational (rule-based) activities	- Requires the application of learnt mathematical skills, e.g. solving equations, simplifying expressions, collecting like terms, factorizing, substituting, and hypothesizing, mathematical reasoning and computations. - Requires procedural fluency, i.e. one needs to be effective and efficient in using procedures.
Justifying and generalizing activities	- Involves examining, analysing and interpreting representations/models made earlier to be sure they fit the context or to have a deeper understanding of underlying mathematical structure - Involves ascertaining that for the results to be generalized, one must be willing to test the formulas generated.

Source: Adapted from Chan and Foong, (2013a) and Kilpatrick et al. (2001).

It should be noted that the description of mathematical competence, as argued by Kilpatrick et al. (2001), seems to align with the expectations of the South African school curriculum, except for the application of mathematics in real life situations. In their current form, algebraic activities fail to account for the acquisition of linguistic competence, which seem to be a key component of mathematical problem solving, particularly in financial mathematics, as highlighted in the NSC National Diagnostic Reports for the period 2011 – 2017. Further, algebraic activities seem not accommodate the contextual competences necessary for the application of mathematics in other contexts. The conversation that took place among the T_group members when solving Problem 3 underscored the importance of understanding mathematical terms as a critical step in mathematical problem solving (see excerpt 1).

Excerpt 1: T_group Problem 3 Transcript

⋮	
Mkhize:	A debt of R200 000 bearing interest at 5% p.a. compounded semi-annually will be retired by R50 000 in two years plus a sequence of 8 equal semi-annual payments starting at the end of third year. Find the regular payment. Ayi ngeke mfana ayi (No ways boy, no ways).
Moon:	Bo semi-annual (laughs).
Mkhize:	Half-yearly.
Moon:	Oooh half yearly, okay.
Mkhize:	Every six months.
Moon:	Semi.
Mkhize:	Means half.
Moon:	Oooh yaya ya, ake siphinde sifunde mfethu, yini le? Isikweledu angithi? Kusho kuthi lo (A interjects) (oh yes, let's read again, what is this? A debt right? This means (A interjects).
⋮	

Excerpt 1 highlights the dilemma of solving mathematical problems in a second language. It is evident that the PSMTs are grappling with the meaning of the words. This shows that PSMTs have to overcome the language barrier prior to understanding and overcoming the mathematical demands of the problem(s). While Kilpatrick et al. (2001) expounded the latter, they were silent on how a problem-solver gets acquainted with mathematical language.

Polya (2014) acknowledged the importance of language in problem solving by arguing that problem solvers should understand verbal statements. He argued that this would enable the problem solver to make appropriate connections between the data, the unknown and the problem conditions, or to relate the current problem to similar problems encountered.

The South African school curriculum for mathematics defines mathematics as a language and argues that learning mathematicians should communicate their mathematical ideas using verbal modes to describe relationships. To cater for the correct usage of mathematical language, as demanded by the Department of Basic Education (2011a), this researcher deemed it necessary to have mathematical competence, which incorporates all aspects of learning a language (defining terms, conventions, notations etc.) and termed this **linguistic activities**.

The work of Pournara (2013c), on using timelines in the teaching and learning of financial mathematics, stresses the importance of contextual competences (such as knowledge of conventions particularly the t-notation), when solving problems, to clearly indicate the timing (beginning or end of period) of the deposits or payments. In support, Xudong and Mingxing (2014) posited that solving problems in financial mathematics requires problem solvers to identify the number of cash flows and their timing.

Jalbert, Jalbert, and Chan (2004) added that learning mathematicians experience difficulties in identifying the appropriate time value of money (TVM) while solving problems in FM. Solving mathematical problems in financial mathematics is completely different to solving problems in other areas of mathematics because financial mathematics is time dependent and the problem solver has to deal with time reliant variables, such as interest rate, number of payment/deposits and periods, present value and future value. In financial mathematics, problem solvers need to know mathematics and financial principles, such as TVM and compounding of interest, as well as contextual financial conventions (e.g., South African markets use 365 days, while US markets use 360 days). These aspects of mathematical problem solving in financial mathematics do not seem to be catered for in the framework of mathematical proficiency, as expounded by Kilpatrick et al. (2001), and consequently are missing in their algebraic activities as well.

To fully explore PSMTs problem-solving strategies in financial mathematics, this researcher coined the term **contextual activities** to cater for the missing aspects of mathematical proficiency. Hence, contextual activities entail knowledge of financial principles, such as the TVM principle; ways of calculating interests and financial conventions, such as the use of 360 or 365 days in financial markets; and knowledge of current financial practices, amongst others.

In this study, five algebraic activities, instead of three, emanating from the work of Kilpatrick et al. (2001), have been utilised to explore the PSMTs non-routine mathematical problem-solving abilities. These five constructs were critical in this study because they enabled this researcher to examine the mathematical knowledge that supported or hindered the PSMTs problem-solving strategies in FM and to describe the causes of the errors in their solutions.

3.3 Mathematical Problem Solving

3.3.1 What is a problem?

Aljaberi and Gheith (2016) defined a problem as a gap which separates the individual from accomplishing a desired goal. Posamentier and Krulik (2008) defined a problem as a situation that confronts a person, that requires resolution and for which the path to the solution is not immediately known. Näveri et al. (2011, p. 162) posited that a problem is a "task which requires an individual to combine the existing knowledge in a new way (new at least to the problem solver)". Therefore, a problem is a new situation that acts as an obstacle and requires one to embark on an inquiry or investigation to attain the desired goal. Avcu and Avcu (2010) added that for a situation to be a problem, a person must be aware of it, be willing to solve it but be unable to apply existing knowledge directly. It seems a problem is dependent on what an individual knows and what he or she is unable to do in that moment. Rodriguez, Bain, Hux, and Towns (2018) added that a genuine problem is a situation which a problem solver has no experience of and attempting to replicate a teacher's method would not be useful.

In mathematics, problems are classified as routine problems or non-routine[12] problems (Marchis, 2012). The former are dominant in school textbooks, they have a structure and indicate how they could be solved (Aljaberi & Gheith, 2016). These problems require an individual to repeat a learned procedure or algorithm. In solving routine problems, the solution is always guaranteed, one only needs to be systematic and to dedicate time. Non-routine mathematical problems, alternatively, do not hold any indication of how they could be solved. They are open to being solved using many methods (Posamentier & Krulik, 2008). Solving non-routine problems requires one to be flexible. However, there is no guarantee of coming to a solution.

3.3.2 What is problem solving?

Problem solving is an activity with which one engages to change an undesirable situation, including daily activities such as communication, learning, planning, working and decision making (Aljaberi & Gheith, 2016). Mairing (2017) defined problem solving as a deliberate thinking process, aimed at attaining an answer to a problem. In addition, Mairing (2017) posited that thinking is an internal process which could only be inferred from external representations, such as written and or verbal language (words, symbols, phrases) and gestures. Problem solving is a way of applying the previously learned concepts and skills in a new context (Tsruda & Lash, 1985). Chapman (2015, p. 20) added that problem solving involves “engaging in a task for which the solution method is not known in advance, finding a way out of a difficulty, a way around an obstacle, attaining an aim which was not immediately attainable”. Problem solving is what one does when one does not know what to do.

This researcher believes that problem solving is critical in mathematics teaching and learning in that it requires one to draw on all strands of mathematical proficiency (Kilpatrick et al., 2001).

According to Nunokawa (2005), mathematical problem solving arises when the problem solver’s mathematical knowledge cannot be applied directly to the situation at hand. As a

[12] These are problems which require the problem solver to combine the existing knowledge (facts, skills, algorithm, method, formula etc.) in a new for their resolution.

result, the problem solver needs to study the situation to transform it or to acquire new insights. If the problem is not resolved, a further exploration of the problem is pursued based on previously obtained information, which is sometimes incorporated into the mathematical knowledge of the problem solver (Nunokawa, 2005).

It seems that learning mathematicians' ways of thinking about a problem and application of mathematical knowledge in unfamiliar situations are more important than answers. It seems that in a problem-solving situation, the process is more important than the final result, as the process gives rise to new knowledge. As a teacher educator, this researcher is interested in the processes which the PSMTs use when confronted with non-routine mathematical problems in financial mathematics. This study affords this researcher an opportunity to look at the PSMTs' thinking processes and consequently identify the mathematical problem-solving strategies and algebraic activities they use in solving non-routine problem solving in financial mathematics.

Nieuwoudt (2015b) posited that perspectives on problem solving range from traditional approaches to progressive approaches. She further pointed out that, in the former, the teacher is central to teaching and learning and the emphasis is on a mastery of mathematical ideas, skills and procedures. Teaching begins with the teacher demonstrating a skill or algorithm to be learnt, followed by the learners solving a series of exercises, and teaching culminates in learners being given problems to practice their learned ideas and skills (Barnes, 2004). Traditionally, learning mathematics is equivalent to absorption of 'ready-made' knowledge from a teacher. Fauzan (2002) argued that learning mathematics in this manner is anti-didactical inversion, since the way in which mathematicians come to conclusions is reversed.

In contrast, in progressive approaches, learners are at the centre of teaching and learning. Teaching starts with an open-ended problem and learners learn new concepts and skills as they search for a solution to the relevant problem. Kilpatrick et al. (2001) posited that both views are incomplete and fall short in explicating the complexity of mathematics, learning and of teaching. According to them, mathematics is invented and it is discovered. They further argued that people learn mathematics on their own and from others, particularly their teachers, and thus teaching should provide both constrained and open learning opportunities for students.

In solving open-ended problems, problem solving heuristics are used to guide a problem solver and provide a problem solver with ways of tracking the solution process. In progressive approaches, learning mathematicians engage in the mathematical activity of solving problems and teaching means formalizing the learning mathematicians' informal utterances and methods (Fauzan, 2002).The use of problem solving heuristics in mathematical problem solving dates back to George Poyla's seminal work in 1945, when he published a four step heuristic for solving open-ended problems. Since then, numerous models have been developed (see table 4). This study uses Poyla's heuristic to study how PSMTs solve non-routine problems in financial mathematics. This heuristic is further discussed.

Phase 1: Understanding the problem

In this stage, a problem solver seeks to understand the problem by unpacking the verbal statements of the problem (Matlala, 2015). A problem solver should become immersed in the problem and identify the parts of the problem needed in its resolution, namely the givens, the unknowns, the conditions and the goals (Stewart et al., 2017). Problem solvers should translate the problem into other modes of representation, such as graphs, tables, diagrams, and equations. In reading the problem, a problem solver must identify the mathematical concepts involved, their relationships and operations.

Schoenfeld (1985) described this stage as the resource stage, wherein a problem solver mobilizes mathematical facts, algorithms, conventions, intuitive and formal knowledge, which will then inform the solution process. Aineamani and Niacker (2014) argued that at this stage the problem solver must demonstrate a firm grasp of mathematical ideas by using multiple representations (diagrams, equations, graphs and tables) to communicate an understanding of the words in the problem. They further added that "when you have a complex wordy problem … it helps to visualize the data as a diagram; diagrams make the information easier to understand" (Aineamani & Niacker, 2014, p6). Visual representation of a problem goes beyond demonstrating an understanding of the problem in that it affords a problem solver a clear idea of the quantities involved, and thus facilitating the choice of a problem-solving strategy.

Devising a plan

According to Polya (2014), at this stage, problem solvers, in this case PSMTs, make their ideas formal by devising a strategy to be employed to arrive at the desired outcome. Polya (2014) argued that PSMTs have a number of strategies to use.

Kilpatrick et al. (2001) posited that PSMTs need to have a productive disposition to execute these strategies successfully. PSMTs need to show a knowledge of mathematics and have the ability to monitor their progress so that they may make necessary adjustments to their strategy as they gain more knowledge about the problem. The PSMTs would need to be able to question not only their understanding of the problem but also the appropriateness of their strategy.

Carrying out the plan

Kilpatrick et al. (2001) argued that this stage requires PSMTs to be able to carry out procedures flexibly, efficiently, effectively and appropriately. PSMTs need to be able to formulate and represent mathematical problems using other modes of representations, such as equations and expressions, graphs and tables. At this stage, PSMTs are expected to formulate conjectures and justify them, experiment, explain and solve the mathematical problem. Here PSMTs need to be able to monitor their progress while keeping an accurate record for their work. Krulik and Rudnick (1982) and (Polya, 2014) contended that PSMTs should apply a variety of strategies to solve a mathematical problem. However, success in solving mathematical problems is determined by the quality of previously mastered mathematical knowledge and skills held by PSMTs.

Looking back

According to Polya (2014), at this stage, problem solvers are required to interpret the solution, in relation to the problem, to determine the extent to which it meets the problem conditions and whether it is sensible.

Consider the following problem:

> *Three sons inherited five cars from their father which they shared equally.*
> *How many cars does each son get?*

From a mathematics' point of view, sharing translates to division which gives the answer $1\frac{2}{3} cars\ per\ son$. This is senseless in the real world. At this point, one would also check if there are other strategies that could be used in determining the solution.

This researcher believes that successful non-routine mathematical problem solving in financial mathematics is dependent on one's mathematical knowledge, strategic knowledge (knowledge of problem solving heuristics and problem-solving strategies), fluency in algebraic activities and conventions used in mathematics and finance, as well as communicative abilities in the language of mathematics (Kilpatrick et al., 2001; Pournara, 2013b, 2013c, 2014) coupled with problem solvers' abilities to work with others and personal attributes such as persistence, perseverance, motivation, organization and confidence (Matlala, 2015).

Table 4: Mathematical Problem-Solving Heuristics

Polya (1945)	Krulik and Rudnick (1996)	Hohn and Frey (2002)	Zalina (2005) in Tambychick & Meerah (2020)
i) Understanding the problem. **ii) Devising a plan.** **iii) Execute the plan.** **iv) Look back.**	i) Reading and thinking. ii) Analysing and planning. iii) Organizing strategy. iv) Getting the answer. i) Confirmation.	i) State the problem. ii) Options to use. iii) Link to the past. iv) Visual aid. v) Execute your answer. vi) Do check back.	i) Understand the problem. ii) Solving the problem. iii) Stating the answer.

As may be seen in table 4, recent problem-solving heuristics may be reduced to Poyla's four-stage heuristic. These heuristics have received criticism for over simplifying the problem-solving process by linearizing it. It is argued that when confronted with a problem, problem solvers may jump stages, revert back to earlier stages whenever they hit an impasse and work in cycles (Chan & Foong, 2013b).

Näveri et al. (2011) added that mathematical problem solving requires one to be creative. In support, Schoenfeld (1985) remarked that problem solvers should exercise control over the problem solving process by monitoring decisions taken at each step. Heuristics help novice problem solvers to navigate the problem without guaranteeing solutions to the problem or creation of new knowledge (at least to the problem solver), which is the essences of problem solving.

The purpose of this study is neither to critique these heuristics nor to formulate a new heuristic but to identify pre-service teachers' problem-solving strategies. The heuristics, however, form the basis of this study in that they indicate the phases through which pre-service mathematics teachers are likely to pass as they attempt to solve the given problems. In this way, they provide the medium through which one can explore pre-service teachers thinking process to identify the problem-solving strategies preferred by the pre-service mathematics teachers.

In support, Hohn and Frey (2002) posited that non-routine mathematical problem solving is executed through phases. They argued that the first phase is problem representation which is made up of problem translation and problem integration. This phase is followed by solution planning, solution execution and solution monitoring. Hohn and Frey's ideas on problem solving are succinctly captured by Kilpatrick et al. (2001) notion of algebraic activities.

Problem-solving heuristics help problem solvers to organize, monitor and control the solution process. However, these heuristics do not indicate which knowledge resources enable problem solvers to solve a non-routine mathematical problem successfully. Rodriguez et al. (2018) noted that there are two kinds of problem solvers: Expert and novice problem solvers. They argued that novice problem solvers have inadequate knowledge to draw on and focus on surface features, such as algorithms or equations, while expert problem solvers have a large meaningfully organized body of knowledge, which allows them to reason qualitatively about the problem. Problem solving, in general, and problem-solving heuristics, in particular, have been incorporated into school curricula around the globe. South Africa is not an exception in that regard.

The South African School curriculum aims to produce learners who are able *to* "identify and solve problems and make decisions using critical and creative thinking" (Department of Basic Education, 2011b, p. 5) . This is what mathematicians do – identifying and solving problems. The curriculum accentuates learning Mathematics to doing Mathematics, of which solving everyday life [non-routine] problems is a critical component (Fauzan, 2002). This aim aligns with modern developments in teaching and learning mathematics through problem solving.

Aljaberi and Gheith (2016) have argued that problem solving develops and improves logical thinking, as well as decision-making skills. This implies that teaching mathematicians should organize activities and create learning environments that are conducive for learning mathematicians to reinvent mathematics through active participation and collaboration. Problem solving helps to strengthen both content knowledge and pedagogical knowledge as it affords problem solvers time to reflect on both the process and the outcome and also communicate their problem-solving strategies. In support, Abdullah, Ibrahim, Surif, Ali, and Hamzah (2014a) asserted that when engaged in problem solving, learning mathematicians experience the power and usefulness of mathematics. Non-routine problem solving has a built-in uncertainty since it does not always lead to an answer and the problem is open to different approaches. For teaching mathematicians to embrace the uncertainty, they should have a sound conceptual and pedagogical knowledge.

3.3.3 State of problem solving: A brief review of international literature

Internationally, Brehmer, Ryve, and Van Steenbrugge (2016) studied textbooks to see how non-routine mathematical problem solving in the area of calculus is represented in Swedish upper secondary schools. They concluded that mathematical problem solving was not supported by the textbooks, in which only 5.45% of the 5 722 tasks were mathematical problems.

Marchis (2012) argued that, in Romania, teachers do not emphasize non-routine mathematical problem solving since national tests require learners to apply formulas or algorithms.

The situation in Romania and Sweden seems to be similar to what this researcher has observed in most South African textbooks. Problems that are cognitively demanding are often placed at the end of the list and are normally not done by both teaching and learning mathematicians. Such problems are never included in assessment activities. Teaching mathematicians often draw on and use policy documents (e.g. curriculum) and other resources such as textbooks and exam papers to legitimize what they do. Anything that does not appear in these materials, particularly examination papers, is unlikely to be emphasized during teaching. Marchis (2012) added that teaching mathematicians do not solve non-routine mathematical problems because they lack problem-solving strategies and they are not comfortable with the pedagogical demands needed in non-routine mathematical problem-solving activities. Problem solving demands that teaching mathematicians use their mathematical knowledge in ways which may be different to how the knowledge was acquired (Saygılı, 2017a).

After studying 4th graders solving non-routine mathematical problems, Yazgan (2016) concluded that problem-solving strategies explain 84% of problem-solving successes. In Malaysia, Yew and Zamri (2016) noted that final year students, in a B.Sc.Ed. programme, majoring or minoring in mathematics, lacked problem-solving skills.

A study conducted by Abdullah et al. (2014a) on non-routine mathematical problem solving revealed that the in-service and pre-service teachers involved had low problem-solving skills. They pointed out that these maybe due to (1) teachers' inexperience in solving open-ended problems, (2) teachers having no background in mathematics, and (3) mathematics teaching at universities not equipping teachers with problem-solving skills for non-routine mathematical problems as non-routine problems are not emphasized in lectures attended by teachers.

Linguistic skills are essential in solving non-routine mathematical problems. In support, Al-Jamal and Miqdadi (2013) argued that solving contextualized mathematical problems demands that problem solvers transit from the problem text to the mathematical text. They further contended that the transition requires conceptual understanding in that the problem solver must comprehend the text first, which forms the basis for solving the problem mathematically.

The studies mentioned thus far show that problem solving is neither learnt from textbooks nor during teacher training, even though it is a desired and advocated for practice in teaching and learning mathematics at school level. Further, the studies cited show that while problem solving is advocated for in both the school and teacher education curricular, non-routine mathematical problems are vastly underrepresented in learning and teaching support materials. This is a concern since problem solving is known to foster deep learning, which aids in the construction of problem-solving strategies and the retrieval of knowledge in new situations.

Also, the studies by Yew and Zamri (2018) and Abdullah et al. (2014b) have shown that teachers exit initial teacher education programmes without the requisite problem-solving skills. This researcher wonders if this situation applies to South Africa as well. The findings of these studies are in line with this researcher's experiences as a pre-service teacher and as a teaching mathematician that non-routine mathematical problems are never included in assessment activities at school.

As part of the teaching personnel in an initial teacher education (ITE) programme, this researcher felt that there was a need to explore the PSMTs' problem-solving strategies in the context of solving non-routine problems in financial mathematics, to optimize their learning experiences so that they can embrace problem solving in their classrooms.

Yew and Zamri (2018) also examined pre-service teachers. However, they studied pre-service teachers solving non-routine mathematical problems individually and their problems were based on calculus (maximizing area), whereas in this study prospective teachers work in groups to solve non-routine problems in financial mathematics.

Englard (2010) argued that young children can come up with problem-solving strategies which incorporate or model the actions and relationships in the problem. This seems to gradually fade as children progress through the grades to a point where they start looking for cues, operations just taught, key words and numbers in the problem. Learning mathematicians who solve non-routine mathematical problems tend to attain better scores in tests, national assessments and international assessments, such as TIMSS and PISA (Grønmo et al., 2015; Mullis et al., 2012; OECD, 2014).

In agreement, Abdullah et al. (2014a) posited that teaching and learning should involve problem-solving skills that are robust covering the whole curriculum. Engaging learning mathematicians in non-routine mathematical problem solving allows learning mathematicians to attain mathematical proficiency since their previous learning experiences and contributions are discussed, interrogated and integrated into new learning (Barnes, 2004; Gravemeijer & Doorman, 1999; Kilpatrick et al., 2001).

In South Africa, it has been found that learning mathematicians lack problem-solving skills (Khalo & Bayaga, 2014, 2015) in financial mathematics. This causes them to commit errors when solving problems. Other studies show that both in-service and pre-service teachers are also deficient in problem solving (Makonye, 2017; Pournara, 2013a, 2013b, 2013c). However, Nieuwoudt (2015a) found that Grade 4 learners were able to solve mathematical problems using a variety of strategies. Similar findings were made by Feza (2014a), who argued that South African 9th graders perform better on tasks that require reasoning than those that demand an application and recall of mathematical knowledge, although she acknowledged that their overall performance in mathematics is far below the international mean of 500 points.

There seems to be disagreement amongst South African researchers about the problem-solving abilities of learning mathematicians. One can only surmise that this disagreement is an indication of the unequal distribution of learning opportunities provided by teaching mathematicians or a sign of improvements in the educational system. It is this disagreement that prompted this researcher to explore the problem-solving strategies used by PSMTs in solving non-routine financial mathematics problems. Further, it seems that more research has been done at school level than on pre-service programmes. This researcher is of the view that studying PSMTs still undergoing training would reveal more information about their current problem-solving experiences prior to being constrained by the systemic pressures of the school curriculum, officials or learning mathematicians.

There is consensus among researchers in mathematics education (Aljaberi & Gheith, 2016; Karatas & Baki, 2013; Marchis, 2012; Nieuwoudt, 2015b) that problem solving is critical to teaching and learning mathematics at school. This researcher embarked on this investigation to find out if South African PSMTs have different or similar learning experiences of non-routine mathematical problem solving to their counterparts in other countries. Such knowledge could be used to restructure mathematics education courses or modules and ultimately influence how mathematics is learnt at school.

3.3.4 Mathematical problem-solving strategies

A mathematical problem is a situation that demands the use of mathematical knowledge (concepts, skills, procedure or algorithms, and mathematical practices) to resolve it. According to Montague and Bos (1990), the cognitive and metacognitive processes of applying mathematical knowledge to resolve a mathematical problem is called mathematical problem solving. Despite an advocacy for incorporating mathematical problem solving into school curricula by researchers in mathematics education and policy makers, problem solving has not been successfully realized in schools (Posamentier & Krulik, 2008). Posamentier and Krulik (2008) attributed this to the weakness of the training which teaching mathematicians received in problem solving. They argued that problem solving can be seen as (1) a subject of study in and of itself (i.e. problem solving heuristics and strategies are taught and learnt), (2) as an approach to a particular problem (i.e. an attempt to resolve an undesirable situation) or (3) as a way of teaching (i.e. mathematics is taught through problem solving).

In this study, one focuses on mathematical problem solving as a way of teaching with a particular reference to the problem-solving strategies used by final year pre-service mathematics teachers. There are a number of problem-solving strategies which a teaching mathematician may call upon during teaching. These strategies may be used alone, or in combinations, to solve a problem. Researchers in mathematics education have identified the following strategies (see table 5), in no particular order, as those that are mostly employed in mathematical problem solving.

In this study, PSMTs were given problems similar to problem 2:

<u>*Problem 2*</u>

You open a NedTerm account with Nedbank in April 2008. This account allows you to make monthly deposit. When you open the account you make an initial deposit of at least R1000.00 and then monthly payments of R100 or more. You decided to deposit initial amount of R1000.00 and monthly deposit of R300. These payments are made at the start of every month and you do this for 18 months. The monthly payments start in May 2008. The interest rate is 7.65% p.a. compounded monthly. How much money will you accumulate over this period?

In order to resolve problem 2, PSMTs could break the problem into two separate problems (**reduction or solve a simpler problem strategy**) by first focusing on what happens to the R1000.00 as time passes on. This would enable them to either perform repetitive calculations (use the **compute** strategy) over the 18 months or use a **formula** to predict the expected amount after 18 months. PSMTs would then deal with the R300.00 separately, i.e. carry out computations for each R300 for the duration of the investment, or make a single computation for all of them by using a formula. PSMTs would then combine the answers to respond to the given question.

However, it should be noted that breaking the problem like this does not guarantee that PSMTs would get the correct answer. This is because, unlike in other branches of mathematics, PSMTs still have to make a decision on when the R1000.00 is deposited and whether it earns interest for 18 or 19 months. These kinds of questions would not arise in other branches of mathematics. It should be noted that due to their specificity, the strategies in table 5 were used as codes in this study (see section 4.4).

Table 5: Problem-Solving Strategies

Problem solving Strategy	Description
Computing or simplifying (CS)	Includes straight application of arithmetic rules, order of operations and other procedures/algorithms.
Using a formula or writing equation (FE)	Entails substituting values into a formula, or expression, selecting the proper formula or formulating equations.
Organizing data: • **Making a table (T).** • **Making a list (L).** • **Making a chart (C).** • **Grouping.**	Involves ordering data so that one can easily: • Identify and interpret patterns and relationships. • Select or pick the correct answer.
Making a model or drawing a diagram	Includes: • Writing equations. • Using objects. • Using visual representations with all relevant information labelled.
Intelligent guessing and testing (Trial-and-error)/ systematic guessing (IGT or SG)	Involves: • Making a reasonable guess, testing the guess and revising the guess if necessary. • Making list or tables as a way of organizing the trials it needs a systematic approach.
Solving a simpler equivalent case (SC) or reduction (R)	It demands maintaining the structure of the problem, including: • Rewording the problem. • Using smaller numbers. • Using a more familiar setting. • Dividing the problem into simpler problems.
Elimination (E)of possible situations	It involves eliminating possible solutions based on information presented in the problem or elimination of incorrect answers.
Looking for a pattern (LP)	Involves determining a common characteristic that can be generalized
Working backward (WB)	• Identifying the end and the beginning of the problem. • Reversing all operations. • May use tables and lists to organize the steps.
Logical reasoning (LR)	• Making a series of inferences based on the information presented in the problem. • Mathematical operations are applied. • *Require reading between the lines.*
Adopting a different point of view (AD)	Requires one to find the most elegant way of solving the problem. This should be the method that is less cumbersome but most efficient.
Extreme case (EC)	Involve calculating extreme values.

Adapted from (Krulik & Rudnick, 1982; Mabilangan et al., 2012; Montague & Bos, 1990; Posamentier & Krulik, 2008; Saygılı, 2017b; Yew & Zamri, 2016).

These strategies constitute the essential strategies for mathematicians and mathematics users (Ofori-Kusi, 2017). The question for this study is whether pre-service teachers use these strategies, or others, when engaging in non-routine mathematical problem solving in financial mathematics.

Kolovou et al. (2011) argued that these strategies exert varying cognitive demands on the problem solver, and hence each has different effects on problem-solving success. Intelligent guessing and testing, commonly called trial-and-error, for example, is commonly used in classrooms and everyday life and is known to have low cognitive demands. Saygılı (2017b) pointed out that there has always been a debate within the mathematics education community as to whether these strategies could be taught to students or not, or how to teach them.

According to Mabilangan et al. (2012), strategies used in problem solving are not explicitly taught by teaching mathematicians. They argued that problem solving requires several strategies to be deployed and that the strategies taught by teachers are limited to problems found in textbooks (which are known to be routine problems), as these are dominant in classrooms. Problem-solving instruction in mathematics classrooms is aimed at affording learning mathematicians opportunities to think critically, analyse situations and apply appropriate methods, algorithms and mathematical knowledge to resolve challenging situations (Saygili, 2017).

3.4 Mathematical Problem Solving in Financial Mathematics: Two Approaches

Solving mathematical problems, just like everyday problems, requires one to have a method (a way of doing) and a tactic (a strategy of doing). According to Pournara (2009), mathematical problem solving in financial mathematics can be done by following either an account balance (AB) approach or an individual payment (IP) approach. These approaches are not completely demarcated and may be used sequentially in solving a problem. They both require the problem solver to mathematise a scenario. The AB approach requires that the problem solver tracks the balance in the account at the end of the compounding period. The AB approach imitates the actual practice of banks in that deposits are added to the balance, and the account is adjusted for further withdrawals or deposits each day, while interest is calculated daily, but it is converted to the principal at the end of every month. Due to its intuitiveness, the AB approach engages learning mathematicians in numerical problem-solving strategies where they perform simple iterative calculations until the answer is obtained. This approach may, however, be cumbersome and tedious when dealing with many investment intervals. This researcher has observed that the AB approach is seldom used in LTSMs, although it concretizes the problem-solving process.

In contrast, the IP approach tracks an individual payment throughout the investment term. This approach assumes that each deposit is done in isolation of the other deposits. The value of the investment is equal to the sum of the individual smaller investments. Since each deposit is invested on compound interest, the IP approach relates investment over time to geometric series. The IP approach, therefore, enables the problem solver to generalize through modelling. Mathematical models, such as formulae, enable the problem solver to be time savvy and more accurate, since answers can be attained though substitutions. Due to its ability to generate mathematical formulae, Pournara (2009) stated that the IP approach provides mathematical structure and allows for adjustments in the case of irregular payment plans, such as missing a payment or over payment.

Experience in solving problems in FM is a necessary exercise for solving financial problems at a later stage. However, Parramore (2011) and Pournara (2013a) concurred that the teaching of FM sometimes does not align with the practices of financial institutions. For example, in teaching, deposits or withdrawals are assumed to occur either at the beginning or at the end of a month, a quarter, or a year, and not at any time, and thus give the impression that interest calculations are based on whole years or at least whole discounting/compounding periods. This is a practice that can be considered in contrast with what actually happens in the banks, where interest is calculated daily and compounded at regular intervals.

Despite that limitation, the strategies learnt at school may prove useful when one deals with personal finance issues, such as credit, saving, investment and spending (Gunawan

& Koto, 2017). A knowledge of problem-solving strategies in FM can help one to make mathematically grounded decisions about one's finances. According to Joshi (2019), problem solving in FM is dependent on understanding the time value of money (TVM) principle and the importance of interest in financial transactions. The TVM principle refers to the understanding that money in hand is worth more than the same money received at a later stage due to its capacity to earn interest or inflation.

According to Parramore (2011), solving mathematical problems in FM requires one to thoroughly understand the issues, and the ability **to formulate appropriate mathematical models** to solve the problem and interpret the output of such models within the problem context, as well as being able to make public any **simplifying assumptions** employed in the solution process.

While the research cited in this section supports these approaches and strategies as necessary, it does not analyse whether these are learnt at school or in initial teacher education (ITE). Hence, a study which explores PSMTs' learning experiences in solving non-routine mathematical problems in FM, as they exit initial teacher education, is of outermost importance not only for improving teacher education programmes but also for any country that seeks to improve learning achievements in mathematics. Without knowing where and how problem-solving strategies are learnt it would be difficult to make necessary advancements in resolving perennial underachievement in mathematics.

3.5 Opportunities to Learn Mathematical Problem Solving Through Textbooks

According to Hadar (2017), textbooks are major conveyors of curricula. Textbooks link the policy (intended curriculum) and the classroom instructional activities (implemented curriculum) of teaching mathematicians. Textbooks are a resource for both learning and teaching mathematicians in a learning process (Hadar & Ruby, 2019). Textbooks control not only what is learned but also how it is learnt and taught (Hadar, 2017). Textbooks can constrain or broaden learning mathematicians opportunities to learn problem-solving strategies (Gracin, 2018). The problem-solving strategies which are emphasized in textbooks influence the problem-solving strategies which teaching mathematicians use

(Wijaya et al., 2015). Hence, textbooks have a pedagogical influence which may discourage or encourage the use of different strategies.

Kolovou et al. (2011) pointed out that repeated use of a problem-solving strategy entrenches such a strategy at the expense of other strategies. This implies that teaching and learning mathematicians' problem-solving experiences can be limited by what textbooks offer. Limited exposure to problem-solving strategies has an influence on mathematical attainments of learning mathematicians (Kolovou et al., 2011; Nussbaumer, Schneider, & Stern, 2014; Yazgan, 2016). A textbook selected by a teacher or a school can influence what is learned, how it is learned and the extent to which it is learned (Hadar, 2017; Mkhatshwa & Doerr, 2016). Gracin (2018, p. 1003) argued that the use of textbooks is inseparable from instruction in mathematics:

> *Textbooks play an important role in mathematics education … and are used to a great extent in mathematics classrooms … They are regarded as artefacts that translate policy into pedagogy and represent a link between the intended and implemented curriculum; they reflect the potentially implemented curriculum … Textbooks are considered to be the most frequently used resources in lesson preparation in some countries, even more so than the curriculum outlines.*

Seemingly, textbooks play a critical role in determining the kind and number of problem-solving strategies which learning mathematicians come to know and use. Hadar and Ruby (2019) expounded that in a country that has a centralized national curriculum, where LTSMs are state authorized, textbooks reflect the nation's values and aspirations, as well as what governmental authorities legitimize as the necessary mathematical knowledge to be acquired. This researcher has indicated earlier that NSC – CAPS envisages learning mathematicians who are able to solve mathematical problems without specifying which strategies should be used. The issue, therefore, is which problem-solving strategies are learning mathematicians being exposed to by the teaching mathematician? It is not the aim of this study to analyse the extent to which textbooks offer learning mathematics or the PSMTs opportunities to learn mathematical problem solving but one draws on the research that has been done on opportunities to learn mathematical problem solving offered by textbooks to determine whether the PSMTs learning experiences in mathematical problem solving could be the result of their past learning experiences.

3.5.1 Opportunities to learn mathematical problem solving through textbooks: A South African perspective

Mathematical problem solving is pivotal in many modern daily-life activities, such as employment, further studies, participation in politics or economic activities of the society. As a result, the mathematical achievements of learning mathematicians have taken a pivitol role, at a national level, in many countries (Hadar, 2017; Wijaya et al., 2015). The ability to solve non-routine problems has become an educational goal (Wijaya et al., 2015). In line with this goal, each nation develops its own curriculum, and thus setting its minimum performance standards that must be attained by all learning mathematicians. The curriculum specifies the content (knowledge, skills, values and attitudes) to be acquired at each stage of the learning process and the instructional pedagogy to be used. In South Africa, the curriculum advocates for problem solving and its accompanying learning and teaching support materials (LTSM) in various forms (textbooks, learner and teacher guides, charts, exam papers etc.), which may be in either print or electronic formats, have been developed accordingly. The LTSM recontextualises the curriculum (Hadar & Ruby, 2019).

A recontextualisation of a curriculum is dependent on the authors' interpretations of the curriculum policy documents which may differ with the teachers' interpretation. To mitigate the interpretation dilemma, all textbooks used in South African school are pre-approved by the Department of Basic Education. Among the criteria used is the alignment of textbooks to NCS – CAPS. Thus, textbooks are expected to enculturate learning mathematicians into mathematical activities – one of which is mathematical problem solving. The South African Department of Basic Education requires that learning mathematicians should develop mathematical practices (investigate, represent, analyse, interpret etc.) by engaging in non-routine problem solving whilst at school (Chirinda & Barmby, 2018).

An electronic search of scholarly articles in the period 2011 – 2019 with key terms 'problem solving in textbooks' yielded one article on South African Grade 9 teachers' views on the teaching of mathematical problem solving by (Chirinda & Barmby, 2018). No study regarding mathematical problem solving in textbooks was published during this period in SA although the final NCS – CAPS document and its LTSMs were published and implemented in 2011. Further, this implies that the problem-solving abilities of post 2011 school leavers, or opportunities to learn mathematical problem solving offered to learning mathematicians by LTSMs, have not been investigated in SA within this period. Considering this, the importance of this study cannot be overemphasized as it explores the problem-solving strategies used by the PSMTs, who are mostly post 2011 school leavers, specializing in mathematics teaching, as they exit initial teacher education. Chirinda and Barmby (2018) reported on the views of teaching mathematicians about teaching mathematical problem solving at the expense of problem-solving strategies (which this researcher deems critical), which were used by the teaching mathematicians and learning mathematicians during the teaching and learning process.

While Chirinda and Barmby's (2018) study involved in-service teachers, their findings are critical to this study and are discussed since this study also emanates from a South African context. After gathering the views of 31 teaching mathematicians through an open-ended questionnaire in one district in Gauteng, Chirinda and Barmby (2018) observed two of these teachers over an extended period. They concluded that participants had a partial understanding of what constitutes mathematical problem solving since they believed that teaching mathematical problem solving was demonstrating examples to learning mathematicians and giving them practice exercises on which to work. In addition, they argued that most participants' views and practices of mathematical problem solving can best be described as the traditional approach of teaching of mathematics. In the lessons observed, in their study, there were no problematisation of mathematics. In those classrooms, learning mathematics was limited to mimicking and recognizing, rather than engaging in creative and reflective activities. The participants viewed engaging learning mathematicians in mathematical problem solving as a time consuming extra activity and not as the major goal of teaching mathematics.

Chirinda and Barmby (2018) stated most participants (28 out of 31) indicated that they use prescribed textbooks for planning and teaching. Chirinda and Barmby (2018, p. 120) further posited that the "*DBE textbooks have few non-routine problems at the end of each* *chapter but we realised that teachers avoided these problems*". This implies that the DBE approved textbooks offered learning mathematicians limited opportunities to learn mathematical problem solving while at school.

Non-routine mathematical problem solving has become a global educational goal for teaching mathematics as it enables learning mathematicians to reinvent mathematics by engaging in mathematical practices such as investigation, making and justifying conjectures, and observation. Seemingly, the attainment of this global educational goal solely rests in the hands of teaching mathematician as very few textbooks have non-routine mathematical problems. It is essential that teacher educators examine the kind and quality of problem-solving strategies used by the PSMTs when confronted by non-routine mathematical problems before they exit teacher education. This is critical because the strategies employed by the PSMTs are more likely to be used in the classroom. This would afford teacher educators an opportunity to optimize the PSMTs' learning experiences to ensure global competitiveness.

South Africa is an active part of the international community and its neighbouring countries. It is therefore essential to compare opportunities to learn mathematical problem solving offered by South African textbooks to those found in textbooks from other countries. In the next section, the research about opportunities to learn mathematical problem solving offered to learning mathematicians by textbooks found within the international community are examined.

3.5.2 Opportunities to learn mathematical problem solving through textbooks: Global perspectives

Mkhatshwa and Doerr (2016) studied the business calculus textbooks used by undergraduate business and/or economics students in the United States to determine whether the textbooks offered the students opportunities to learn mathematical problem solving. They concluded that the textbooks do not provide students with worthwhile and adequate non-routine mathematical problems to acquire the problem-solving strategies necessary for partaking in a business and/or economic context.

After analysing the textbooks used by fourth and eighth graders in Israel, Hadar and Ruby (2019) concluded that the textbooks used in Israel place varying cognitive demands on learning mathematicians. Learning mathematicians using a textbook which demands the use of higher order thinking skills (HOTS) when solving problems tend to have higher scores in mathematics assessments (Van Zanten & van den Heuvel-Panhuizen, 2018). Thus, textbooks become a potential source of inequities in learning mathematicians' opportunities to learn mathematical problem-solving strategies.

Van Zanten and van den Heuvel-Panhuizen (2018) conducted a follow up study to that of Marja van den Heuvel-Panhuizen and Kolovou (2009) on the textbooks used in Dutch primary schools. They concluded that the proportion of non-routine problems included in Dutch primary school textbooks had dropped from 9%, in 2009, to 6%, in 2018. In addition, Van Zanten and van den Heuvel-Panhuizen (2018) found that most non-routine problems were included in additional enrichment materials.

Wijaya et al. (2015) found that only 2% of the problems in Indonesian textbooks for ninth and tenth graders were non-routine problems. After analysing 22 000 tasks from sixth, seventh and eighth grade textbooks, Gracin (2018) found that non-routine problems were underrepresented in Croatian textbooks. Similarly, after analysing 3 535 problems from two Spanish publishers, Vincente Vicente, Manchado, and Verschaffel (2018) concluded that textbooks do not afford learning mathematicians adequate opportunities to learn mathematical problem solving and recommended that teaching mathematicians be empowered so that they can supplement textbooks as they mediate learning.

In Sweden, Brehmer et al. (2016) posited that of the 5 722 tasks they analysed from various textbooks, only 312 (about 5.45%) were classified as non-routine mathematical problems. They further indicated that of the 312 tasks categorized as mathematical problems, 264 (about 84.62%) of them were found at the end of the chapter. In this researcher's view, the placement of non-routine mathematical problems at the end of a chapter relegates them to 'add-ons' (activities which teachers may or may not do), and not contributing to the essence of learning mathematics.

The studies cited thus far have indicated that school textbooks in their current form do not support the learning of non-routine mathematical problem solving and the acquisition of problem-solving strategies in particular. Hence, it is essential to find out if mathematical problem solving in general and problem-solving strategies are central in post school learning activities and textbooks.

3.5.3 Opportunities to learn mathematical problem solving through LTSMs designed for pre-service mathematics teachers

McCrory and Stylianides (2014) analysed 16 mathematics textbooks in print in the USA, designed for prospective elementary (Grades 1–5 or 6, being ages 6–11 years old) teachers to determine the extent to which these textbooks offer those teachers the opportunities to learn mathematical problem solving, in particular reasoning and proving. They found that this aspect has not been explicitly addressed in textbooks.

After studying a number of mathematical problems across universities in South Africa for courses designed for teachers, Adler and Davis (2006) concluded that the courses did not offer teachers adequate opportunities to learn mathematics in ways that would enhance the teaching of mathematics. They posited that the courses emphasized the compression of mathematical ideas instead of 'unpacking'. 'Unpacking' refers to learning mathematics in its unrefined form. In this study, unpacking would be synonymous to solving non-routine problems in that it allows for usage of informal strategies in solving problems. The lack of mathematical problem solving in LTSMs, as well as in assessment tasks offered to teachers, may indicate an implementation dilemma. The cited studies have indicated that non-routine mathematical problem solving, and consequently problem-solving strategies, are not adequately addressed in most of the textbooks used at school level and the ITE programmes across nations. The question that remains is where are mathematical problem solving, in general, and problem-solving strategies learnt if they are not prioritized in school textbooks?

The underrepresentation of non-routine mathematical problems in the textbooks of the countries cited seems to be a common trend in countries which perform at or below the international average of 500 points in international assessments, such as TIMSS. The situation in South Africa, Sweden and other underperforming countries is a direct opposite to what happens in Singapore, which has consistently been classified as a top performer in international assessments, such TIMSS and PISA. Leong et al. (2011) argued that in Singapore non-routine mathematical problem solving is central to learning mathematics and teacher education efforts are geared towards realizing this goal. Non-routine mathematical problem solving is the key motivator and impetus of learning and teaching in China, Singapore, Korea, Japan and Russia. These countries are classified as consistent top performers in international assessments (Marks et al., 2019; Mullis et al., 2012; Reddy et al., 2013; Reddy et al., 2016). In support, Cai and Jiang (2017) posited that Chinese textbooks have more non-routine mathematical problem solving tasks than USA textbooks.

According to Takahashi (2016), mathematical problem solving has been at the centre of Japanese teaching of mathematics since 1960s. By approving textbooks that introduce new concepts and procedures through a problem, the Japanese government supports teachers in teaching mathematics through problem solving. The textbooks include more alternative problem-solving strategies to a problem and provide diagrams for use by learning and teaching mathematicians (Takahashi, 2016). Similarly, Leong et al. (2011) posited that in-service teachers in Singapore take short courses of about 12 hours as part of their professional development (PD) annually, while those registered for MEd. take a 39 hour module on non-routine mathematical problem solving. The kind of support initiatives offered to teachers to develop their pedagogical knowledge as professionals in China is absent in South Africa. In South Africa, the focus is largely on supplementing the subject matter knowledge of teachers, rather than their pedagogical knowledge, which this researcher deems critical in implementing a problem-solving approach in mathematics teaching.

It is apparent that mathematical problem solving, while it is a desired curriculum goal, it is not prioritized in LTSMs in many countries. The textbook problems in many countries predominantly require solving routine problems aimed at perfecting lower order thinking skills, such as computational skills, rather than HOTS, as envisaged in intended curricula. It is this researcher's view that by limiting non-routine mathematical problems in the textbooks, textbook authors acknowledge that teachers cannot be replaced or surpassed. Teachers are the ultimate arbiters in the learning process.

In justifying the underrepresentation of non-routine mathematical problems in textbooks, Gracin (2018) argued that routine problems need less time to solve while non-routine problems are difficult to design. Gracin (2018) further posited that the omission of non-routine mathematical problems in textbooks indicates that textbook authors' traditional view of mathematics and teaching, which emphasize fluency in low order thinking skills, such a fluency in computation, rather than HOTS. It seems teaching mathematicians, who work with a reform agenda in mind, would have to modify mathematical problems in textbooks to get the best out of their learning mathematicians. A textbook acts as a knowledgeable other in the learning of mathematics just like a teaching mathematicians (Aineamani & Naicker, 2014). Therefore, textbooks should provide learning opportunities to learn mathematical problem solving similar to those expected from the instruction of teaching mathematicians. Hence, the textbooks developed for a reform oriented curricular should fully embrace the intended teaching approach.

The research cited on opportunities to learn mathematical problem solving from textbooks has indicated that the desired goal of teaching mathematics (i.e. to develop problem solving skills in learning mathematicians) is likely to remain an unrealised goal for many nations. This is mainly because mathematical problem solving is taken to be a peripheral issue, as indicated in most textbooks. Aineamani and Naicker (2014) posited that a textbook acts as a knowledgeable other to learning mathematicians, and consequently it should provide opportunities to learn mathematical problem solving, similar to those expected from a teaching mathematician's instruction. To change the current classroom practices which seems to develop lower order thinking skills to reform oriented practices that aim at inculcating higher order thinking skills in learning mathematicians, the mathematical problems given to learning mathematicians should be changed from routine to non-routine problems first (Abdullah et al., 2014b). However, this global goal of teaching mathematics cannot be attained without equipping and skilling teaching mathematicians with mathematical knowledge and problem-solving strategies necessary for embarking on non-routine mathematical problem solving. Similarly, LTSMs and assessment policies which do not align with an advocated teaching approach do not help teaching mathematicians in discovering weaknesses in their instructional activities.

Aineamani and Naicker (2014) argued that textbooks used in the classroom determine how learning mathematicians come to know and view mathematics. Additionally, textbooks can foster or restrict learning mathematicians' opportunities in communicating mathematical ideas, knowledge of various problem-solving strategies, how one solves non-routine mathematical problems and the development of HOTS. The cited research has indicated that there is a need to explicitly incorporate mathematical problem solving in LTSMs and in instructional activities.

In the previous sections, this researcher has drawn on research in mathematics' education to argue that while non-routine mathematical problem solving has not been realized in most classrooms as a way of teaching and learning mathematics, research in mathematics' education and most school curricula advocate for it. The extent to which non-routine mathematical problem solving is incorporated in curriculum policies and classroom activities seems to differ across countries. For example, in South Africa, it is allocated a mere 10% in assessment tasks (see figure 1).

It was further argued that textbooks for learning mathematicians, as well as those meant for PSMTs, do not only underrepresent mathematical problem solving but also relegate it to peripheral activities done at the end of a chapter. This implies that the PSMTs from underperforming countries, such as South Africa, enter teacher education with a limited knowledge of non-routine mathematical problem-solving strategies, although the school curriculum expects them to engage learning mathematicians in the practices of mathematicians, one of which is solving problems.

This researcher further drew on both local and international studies to show that mathematical problem solving has been intensively researched at school level. Previous research examined the models developed by learning mathematicians and the strategies they used in solving non-routine mathematical problems. The studies conducted in South Africa concluded that learning mathematicians have problems in tackling non-routine mathematical problems (Mogari & Lupahla, 2013; Sepeng, 2014; Sepeng & Kunene, 2015; Sepeng & Sigola, 2013).

Mogari and Chirove (2017) expanded upon this point by concluding that success in solving routine mathematical problems has no bearing in solving non-routine mathematical problems. However, the study conducted by Loji (2012) with Engineering 1 students showed that the students who enrolled for this course lacked problem-solving skills, which greatly affected their progress to the second year of their studies. Case studies conducted beyond South Africa on pre-service teacher's problem-solving strategies (Aljaberi, 2015; Aljaberi & Gheith, 2016; Yew & Zamri, 2016) concluded that pre-service teachers lacked problem-solving skills.

This study builds on these works by exploring PSMTs learning experience in solving non-routine mathematical problems in financial mathematics and their strategy flexibility in solving problems. In the section that follows research on strategy flexibility in solving non-routine mathematical problems is discussed.

3.6 Strategy Flexibility

A strategy is a method or a way of finding a solution to a problem. This view is shared by Nussbaumer et al. (2014), who defined a problem-solving strategy as a step by step procedure for solving a problem. Yazgan (2016) argued that problem-solving strategies are techniques used in exploring, analysing and examining different aspects of a non-routine problem to formulate pathways to a solution. Rathgeb-Schnierer and Green (2013) added that a strategy describes the solution process but not how the answer to the problem is determined. They further argued that, to determine the answer, the problem solver should be armed with the tools of the practice, such as concepts, operations and strategies.

Table 5 shows some commonly used problem-solving strategies. The application of problem-solving strategies requires one to be flexible, since strategies that worked in one problem, in one context, may not be directly applied to another problem, in a different context. Kolovou et al. (2011) argued that flexibility in strategy use is negatively influenced by experiencing success in using standard algorithms or procedures in solving routine problems. Standard algorithms are easily entrenched in learning mathematicians and become less flexible. For example, learning mathematicians who have been exposed to the use of the formulae $F_v = P_v(1+i)^n$ tend to apply it every time they see the word 'compound' in the question, irrespective of the problem context and conditions. Taking this point further, DeCaro (2016) argued that inflexible problem solvers tend to overlook certain features of the problem or extract less information from the problem, which affects their mental representation of the problem. This inhibits them from seeing features that could have indicated that they need to modify their strategies.

Alternatively, Star and Rittle-Johnson (2008) posited that strategy flexibility demands that problem solvers have an extensive knowledge of problem-solving strategies to adapt to the new problem situation. Thus, strategy flexibility can enhance the development of conceptual knowledge and may aid knowledge transfer. In support, DeCaro (2016) argued that learning mathematicians who know and use multiple strategies tend to put more information on their mental or physical representation of the problem, which supports the development of relationships between the concepts involved and consequently aids the development of deeper conceptual understanding. It is therefore essential to find out if this current group of PSMTs had a repertoire of problem-solving strategies which they could use flexibly in teaching financial mathematics at school level.

Flexibility in problem solving is the ability to switch between problem-solving strategies during problem solving (Kolovou et al., 2011). Berk, Taber, Gorowara, and Poetzl (2009) argued that flexibility in mathematical problem solving entails a knowledge of multiple problem-solving strategies, an ability to solve the same problem with different strategies and the ability to select strategies which minimize computational steps. In contrast, strategy flexibility in solving non-routine mathematical problems is defined as the knowledge of multiple problem-solving strategies and the ability to choose the most suitable strategy for the given problem. Put differently, a flexible problem solver is a person who is able to switch problem-solving strategies with ease and is able to choose the most appropriate strategy for a particular problem (Newton et al. 2019).

There seems to be no single definition for strategy flexibility amongst scholars since some focus only on one’s ability to switch strategies, while others add appropriateness of the selected strategy in their definitions. However, the researchers have concurred that a problem solver should have a repertoire of problem-solving strategies. This study intends to find out if the cohort of PSMTs have a repertoire of problem-solving strategies. Hence, in this study, strategy flexibility is defined as the ability of a problem solver to change or switch between problem-solving strategies.

Strategy flexibility is an attribute which allows a problem solver to change or adjust strategies during problem solving, depending on information gathered about the problem conditions and problem context. Researchers, such as Xu et al. (2017), argued that strategy flexibility is an important component of mathematical proficiency. Mathematical proficiency in problem solving entails conceptual knowledge, i.e. the ability to comprehend mathematical concepts, operations and relationships and strategic (procedural) knowledge, such as the knowledge of problem-solving strategies and the ability to choose and apply them appropriately in solving a range of mathematical problems in different contexts (Hästö et al. 2019; Kilpatrick et al., 2001; Rittle-Johnson et al., 2012).

The change in strategy used during problem solving could occur as a consequence of one's reflection on the problem solving process, a prompting by a peer or a teaching mathematicians or hitting an impasse during problem solving. Kolovou et al. (2011) opined that switching strategies may take place within a problem, i.e. *intra*-task strategy flexibility or it may occur across problems, i.e. *inter*-task strategy flexibility.

It is the aim of this study to determine the extent to which PSMTs vary their strategies as they solve problems in financial mathematics and determine which strategies are used. Learning mathematicians' low academic performance has been linked to problem solving strategy inflexibility (Kolovou et al., 2011; Star & Rittle-Johnson, 2008). Star and Rittle-Johnson (2008) reported that learning mathematicians fail to adapt their knowledge when confronted with unfamiliar problems. Learning mathematicians' static problem-solving behaviour seems to be in contrast with the well-established problem-solving behaviours of experts, who are able to vary their strategies whenever they experience difficulties during a problem-solving process (Kolovou et al., 2011; Star & Newton, 2009).

Yazgan (2016, p. 100) underscored the importance of problem-solving strategies in solving non-routine mathematical problems by stating that "strategies explain 84% of problem-solving success".

Elaborating further on the importance of strategy flexibility, Star and Rittle-Johnson (2008) posited that learning mathematicians who discover their own problem-solving strategies are more likely to transfer their strategies to other problems and contexts, and they demonstrate deeper conceptual understanding than learning mathematicians who adopt taught algorithms.

It is this researcher's view that to enhance SA learning mathematicians' academic performance in national and international assessments, there is a need to increase their strategic flexibility in solving non-routine mathematical problems. To achieve strategic flexibility in learning mathematicians, the first step is to understand the level and extent to which PSMTs can use and vary their problem-solving strategies, since this is what they should do with their learning mathematicians.

According to Xu et al. (2017), strategy flexibility can be inferred from the problem solver's work. In this study, the researcher analysed the PSMTs' written responses, audio and video transcripts, to identify the strategies used and the extent of strategy flexibility within and across problems by different groups of PSMTs. The methods that have been used in assessing strategy flexibility are (1) choice/no choice method, (2) recognition and evaluation method and (3) generation method (Torbeyns et al., 2009). In the choice method, learning mathematicians/problem solvers are at liberty to choose their preferred strategies for solving the given problem, whereas in no choice method, problem solvers are told which strategy should be used in solving the given problem (Torbeyns et al., 2009; Xu et al., 2017). In the recognition and evaluation method, learning mathematicians are asked to list the steps that would be used to solve the problem and to state which steps would yield the answer more quickly. In the generation method, learning mathematicians are asked to solve the problem several times using different problem-solving strategies. (Kolovou et al., 2011) stated that when problem solvers have experienced success in the use of a strategy, such a strategy gets entrenched, inhibiting the use of other strategies. Torbeyns et al. (2009) added that classroom instruction influences strategy flexibility. This means that the use of multiple strategies from the onset of instruction supports the flexible use of strategies, whereas instruction in one strategy, followed by the use of multiple strategies, inhibits strategy flexibility.

Participants in this study were not novices in learning mathematics or solving problems. They have had instruction in mathematics and have solved mathematical problems (routine or non-routine) at some point. Their past experience was likely to influence how they come to solve problems, as they partook in this study. To overcome the PSMTs inflexibility, this researcher opted to employ the strategy generation method, with prompting, to encourage the PSMTs to use as many strategies as they could think of.

Prompting is known to assist problem solvers in using multiple problem-solving strategies without necessarily increasing the use of efficient strategies (Newton et al., 2019). Other scholars have argued that a knowledge of problem-solving strategies does not imply that problem solvers will automatically use various strategies when solving problems (Kolovou et al., 2011), thus there is a need to activate or trigger the use of multiple problem-solving strategies.

Recent research on non-routine problem-solving strategies examined the problem-solving strategies used by learning mathematicians in various grades across the school curriculum (Mabilangan et al., 2012; Saygılı, 2017b; Sepeng & Kunene, 2015; Yazgan, 2016). These researchers found that learning mathematicians used various strategies in solving non–routine problems. In some instances, learning mathematicians used more than one strategy to solve the same problem. Some researchers have investigated prospective teachers' use of strategies in non-routine mathematical problem solving (Aljaberi, 2015; Aljaberi & Gheith, 2016; Avcu & Avcu, 2010; Bulut & Karamık, 2015) and concluded that the prospective teachers' use of problem-solving strategies was limited and in most cases was accompanied by procedural errors. Given that most of these studies were conducted beyond the borders of South Africa, it is necessary to establish the extent to which South African pre-service mathematics teachers use and vary strategies during problem solving.

In one respect, researchers investigated learning mathematicians strategy flexibility in solving non-routine mathematical problems (DeCaro, 2016; Star & Rittle-Johnson, 2008; Torbeyns et al., 2009; Xu et al., 2017) and concluded that learning mathematicians could vary their strategies, but problem solving context may inhibit flexibility. In contrast, Star and Newton (2009) examined experts' strategy flexibility and found that experts demonstrated strategy flexibility, although they could not choose the most efficient strategy for a given problem.

Considering these international studies, it is debatable whether PSMTs operating in a South African (this researcher defines them as experts) context have an extensive and flexible knowledge of problem-solving strategies in their repertoire as they exit an ITE programme.

The studies mentioned were mainly conducted in the domain of multi-digit numbers and algebra (inclusive of calculus and linear equations) and hence differ from this study, in that this study is located in financial mathematics. The studies which have been examined were conducted in topics whose problems are mainly expressed in symbolic mathematical language, while FM problems are frequently in textual format. Over and above, the studies mentioned investigated the strategy flexibility of individuals, whereas this study explored the strategy flexibility of groups of prospective mathematics teachers.

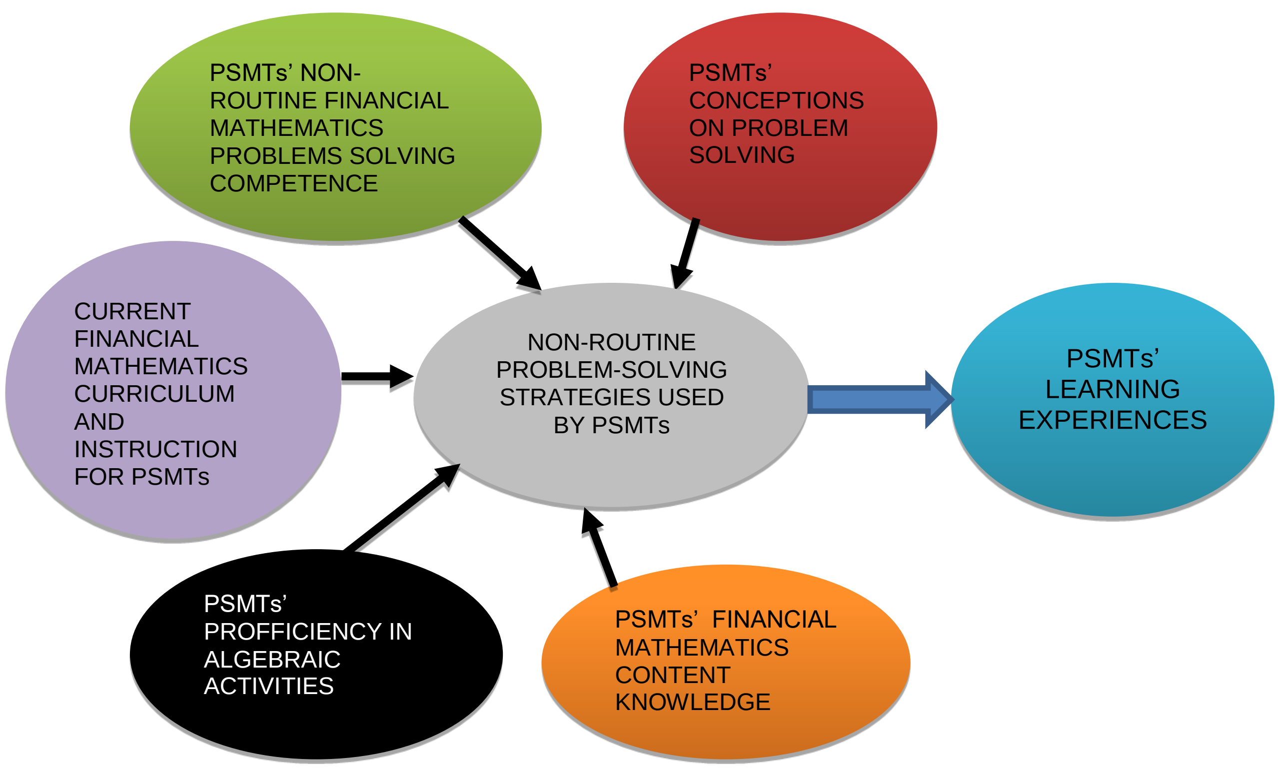
A systematic search of literature using the phrase “strategy flexibility” or “procedural flexibility” shows that no study had been conducted in the period 2007 – 2018 that investigated strategy flexibility among pre-service teachers in FM. This study, therefore, has broadened the current research by exploring PSMTs’ problem-solving strategy flexibility in solving FM problems.

This researcher has noted that FM as a topic within the South African school curriculum has previously been researched though with different intentions Pournara (2008, 2009, 2011, 2013a, 2013b, 2014, 2015a, 2015b) studied a group of prospective mathematics teachers as they received instruction on FM, to identify the mathematical knowledge they needed to know in order to teach FM. Makonye (2017) investigated prospective mathematics teachers’ conception of nominal and effective interest rates, while Khalo and Bayaga (2014, 2015) studied Grade 10 learners to identify the errors made by learners in solving problems in FM. None of these studies reported on the strategies used by learning mathematicians, or prospective mathematics teachers, in solving the problems in FM and the flexibility in the use of such strategies. Hence, it was necessary to determine the kind and quantity of problem-solving strategies that are used by South African PSMTs in solving FM problems. Additionally, the researcher was also interested in whether PSMTs vary their problem-solving strategies within a problem, as well as across problems.

3.7 Overarching construct underpinning PSMTs' learning experiences

In this section, this researcher presents the theories that will be used to explore the phenomenon in relation to the pre-service mathematics teachers' learning experiences. Firstly, an overview of the overarching constructs that constitute the phenomenon, with the relationships among them, is provided. This is then followed by a brief description of the theories involved.

Figure 2: Overarching Constructs



In this study, this researcher explored pre-service mathematics teachers' (PSMTs) choice and use of non-routine problem-solving strategies. To achieve this, the researcher specifically studied problem-solving strategies used by PSMTs when solving non-routine mathematical problems in financial mathematics. This researcher assumed that the problem-solving strategies that PSMTs used were influenced by five factors/constructs (see figure 2). This researcher contended that the way PSMTs did or did not solve problems was a function of their broader 'learning experience' that was acquired and evolved over a long period of peripheral participation in the learning and teaching of mathematics.

It is the researcher's belief that any educational innovations aimed at improving learner performance in mathematics at school level should primarily be directed towards enhancing pre-service mathematics teachers' learning experiences. Hence, this researcher explored the ways in which one can enhance the learning experiences of PSMTs, to make such experiences educationally worthwhile. The exploration has thus focused on PSMT's non-routine mathematical problem-solving strategies. The critical nature of non-routine problem-solving strategies and the associated competence as the single most significant aspect that makes up PSMT's learning experience cannot be overemphasized. Studying the current strategies used by PSMTs was therefore crucial. In the sections that follow, further theoretical elements informing this study are elaborated upon.

3.8 Theoretical framework: Realistic Mathematics Education

This study is rooted in realistic mathematics education (RME) and draws on algebraic activities, as well as problem-solving strategies, to explore the learning experiences of pre-service mathematics teachers in the context of their solving of non-routine mathematical problems in financial mathematics.

The proponents of RME argued that for mathematics to be valuable to humans, mathematics should be useful and play a role in handling societal issues. In support, Marja van Den Heuvel-Panhuizen (2003, p. 9), cited Freudenthal (1977), who posited that "in order to be of human value – *mathematics* must be connected to reality, stay close to children and should be relevant to society" (italics added for emphasis). Consequently, RME distinguishes itself from other theories by requiring the use of realistic contexts and models (representations) in the teaching and learning mathematics. The use of the word 'realistic' is very confusing to many, as it gives the impression that only real-world problems are used in RME. However, Marja van Den Heuvel-Panhuizen (2003) explained that in Dutch, the verb '*zich realiseren*' means 'to imagine'. Hence, the word 'realistic' refers more to the problems or problem situations which problem solvers (in this case

PSMTs) can *imagine* or make sense of or relate to than it refers to the 'realness' or authenticity of problem or problem situations.

In RME, problem solvers are seen as active participants in the teaching-learning process that takes place within the social context of the classroom (Marja van Den Heuvel-Panhuizen, 2003). In this context, problem-solving strategies are generated, shared, challenged and validated. Thus, in RME, problem solvers are offered a learning environment in which they can construct their own mathematical knowledge and have the possibilities of deepening and broadening their understanding of mathematical ideas.

The basis for RME is that learning mathematicians should be afforded an opportunity to reinvent mathematics under the auspices of a teaching mathematician (Barnes, 2004; Fauzan, 2002). Furthermore, formal mathematical knowledge should not be imposed on learning mathematicians but should be the product of formalizing learning mathematicians' informal mathematical knowledge. This can be achieved by allowing leaning mathematicians to problematize mathematics in search of a solution. In support, Hiebert et al. (1996, p. 2) explained that "allowing the subject to be problematic means allowing students to wonder why things are, to enquire, to search for solution and resolve incongruities". Fauzan (2002) posited that learning mathematicians would easily bring their informal knowledge into play if they could solve contextual problems that are realistic[13] to them.

In RME, mathematics is seen as an activity (Fauzan, 2002). Learning mathematics means engaging in mathematical activity, of which solving non-routine mathematical problems is part. According to the Department of Basic Education (2011b), mathematical activities include observing, representing, investigating patterns and quantitative relationships, and making and justifying conjectures, to name a few. These are the practices which mathematicians do: Being able to justify claims, using the symbolic notation efficiently (Rand Study Panel, 2002).

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[13] The term realistic is not limited to problems arising from everyday context only. It includes problems which are experientially real to the learning mathematicians or which learning mathematicians can imagine (Marja van Den Heuvel-Panhuizen, 2003).

Non-routine mathematical problem solving allows learning mathematicians to reinvent mathematical insights, knowledge and procedures by going through mathematization processes (Barnes, 2004). Mathematization is defined as the ability to transit from the world of words to the world of mathematics (Yilmaz & Dede, 2016). Elaborating on this point, Marja van Den Heuvel-Panhuizen (2003) argued that mathematizing means to organize unmathematical matter in a manner that opens it up to being acted upon in a mathematical way (i.e. applying mathematical operations and procedures). Mathematizing is not limited to an application of informal/ intuitive strategies only in solving problems but these forms the basis for the development of formal mathematical procedures and algorithms.

Yilmaz and Dede (2016) identified mathematization competencies which are critical to the problem-solving process as the:

- Ability to identify assumptions necessary to solve the problem.
- Ability to identify variables based on the assumptions made.
- Ability to formulate models based on the assumptions and variables identified.

Yilmaz and Dede (2016) explained that the component ‘identifying assumptions’ relates to the assumptions that are necessary for resolving or for finding a solution to the problem which are compatible with real life; the component ‘identifying variables’ is based on the assumptions that relate to the mathematical variables that are needed for resolving/solving the problem. The component ‘constructing mathematical model(s)’ is based on the relations among the identified variables and the concept ‘relationships’ is related to constructing relations among the variables and constructing mathematical models by benefitting from proper mathematical representations.

Kilpatrick et al. (2001) added that the ability to construct mathematical representations/models of a problem situation is a necessary mathematization competence in solving mathematical problems in general and financial mathematics problems.

Marja van Den Heuvel-Panhuizen (2003) posited that representations or models may be materials, visual sketches, paradigmatic situations, schemes, diagrams and even symbols, but are not limited to these. In this study, timelines are used to depict multiple cash flows (see D_group's solution to problem 2 in appendix H).

Pournara (2013c) argued that in solving financial mathematics problems, PSMTs may use timelines to represent the problem situation (horizontal mathematization) and then apply mathematical means to solve the problem (vertical mathematization). (Department of Basic Education, 2015, 2016, 2017) has argued that during National Senior Certificate Examinations, learning mathematicians draw on mathematical formulae (vertical mathematization) often given in a formula sheet to solve financial mathematics problems, even though they complain that this approach is pitted with misconceptions, misapplications of formulae, and computational and algebraic manipulation errors.

This researcher is of the view that the PSMTs' demonstration of mathematization competences in solving problems in financial mathematics would reveal the robustness of their learning experiences as they exit initial teacher education. Hence, the question that arises for this study is which problem-solving strategies are used by PSMTs as they mathematise the given non-routine mathematical problems in FM?

It is argued that in RME, problem solving occurs through a horizontal and vertical mathematization[14] processes (Aineamani, 2010; Barnes, 2004; Gravemeijer & Doorman, 1999; Marja van Den Heuvel-Panhuizen, 2003). Horizontal mathematization refers to the organising of contexts to make them applicable to mathematical treatment using informal/intuitive knowledge. Horizontal mathematization is characterized by representational activities. Menon (2013, p. 1) added that in "horizontal mathematization, the process of mathematization focuses mainly on ordering and schematising and building a model of the reality so that it becomes amenable to be dealt by mathematical means". For example, PSMTs may use timelines to represent a problem situation. Timelines reduces the amount of noise in the problem by requiring the problem solver to focus only on the relevant and related quantities involved. The use of timelines or any

[14] Mathematization is used in this study interchangeably with non-routine mathematical problem solving to refer to the thinking and reasoning needed in solving problems. "Mathematization is a word that strikes a chord among many, especially those who have been pained by the mechanical manner in which mathematics has been taught and has been used by many to capture what they would consider as the heart of the mathematical enterprise – the thinking and the reasoning" (Menon, 2013, p. 2).

other representation assists the problem solver to transit from everyday text to mathematical text.

Al-Jamal and Miqdadi (2013) added that the transition from everyday text in the problem to mathematical text demands that the problem solver conceptually understands the everyday text, as this form the basis for solving the problem mathematically. For example, when asked show that the square of an odd number is always odd, PSMTs may use **definitions** and **logical reasoning** by arguing that an odd number is a number that ends with either 1, 3, 5, 7 or 9 and that squaring a number that ends with any of these numbers, gives a number that ends with either 1, 3, 5, 7 or 9, and hence the square of an odd number is odd. A response like this indicates horizontal mathematization.

Marja van Den Heuvel-Panhuizen (2003) stated that components of mathematization are intricately intertwined. In horizontal mathematization, learning mathematicians use informal strategies and language to describe and solve non-routine problems, while in vertical mathematisation students use formal procedures and symbolic mathematical language to solve mathematical problems (Aineamani, 2010; Barnes, 2004).Marja van Den Heuvel-Panhuizen (2003) added that in horizontal mathematization, the problem solver moves from the world of life into the world of symbols. The process of selecting (abstracting) essential information and using informal strategies like **acting it out** or **intelligent guessing and testing** would constitute horizontal mathematization, as the problem solver is moving within the world of words.

Alternatively, vertical mathematization refers to the application of formal mathematical means/knowledge to a problem situation through symbolizing and formulation of equations and expressions. In vertical mathematization, symbols are shaped, reshaped and manipulated mechanically, comprehendingly and wistfully (Menon, 2013).

For example, when asked to *show that the square of an odd number is always odd*, PSMTs may respond as follows:

Let k be the odd number

Then $k = 2n + 1, for\ some\ integer\ n$

$$k^2 = (2n + 1)^2$$

$$= 4n^2 + 4n + 1$$

$$= 2(2n^2 + 2n) + 1$$

$$= 2p + 1, for\ some\ integer\ p$$

$\therefore the\ square\ of\ an\ odd\ number\ is\ always\ odd$

Hence, vertical mathematisation is concerned with moving within the abstract world of symbols. It deals with translating the problem (either from its schematic/representation model e.g. timeline diagram) into formal mathematical language by using symbols, operations and choosing an algorithm, such as a formula or equation (Barnes, 2004).

Marja van den Heuvel-Panhuizen and Drijvers (2014) argued that vertical mathematization is a process of reorganizing within the mathematical system by making connections between concepts and strategies. For example, when PSMTs are confronted with a problem like:

> *An investor deposits* $R400$ *every December 15 and June 15 for 10 years in an account that earns interest at the rate of* $8\%\ per\ year, compounded\ semiannually$. *How much will be in the account immediately after the last payment?* (Stewart et al., 2017, p. 868)

To find the solution to this problem, PSMTs would have to undergo the two phases of mathematization. PSMTs would have to decide whether 15 December marks the beginning or the end of the semiannual period (making assumptions about the known and unknown), and hence whether these deposits are made at the beginning or at the end. They would also need to know how many semiannual deposits would have been made in 10 years altogether and whether all deposits earn interest or not. They need to understand the terms used and be able to relate 8% per year to semiannual compounding. Taking the process further, PSMTs could show the series of deposits on a timeline. This process constitutes PSMTs horizontal mathematization since it does not require the use of mathematical means. The next phase of solving this problem would be to decide on the strategies or construct a model to be used in obtaining the answer. Here, PSMTs would need to find the future value of each deposit and when the last deposit is made. This requires the use of a compound interest formula and, to find the sum of all deposits (with interest), PSMTs would need to formulate a geometric series and find its sum. This would constitute vertical mathematization since it requires the use of mathematical means.

Barnes (2004) asserted that mathematics teaching in most South African classrooms has over many years been dominated by traditional and authoritarian approaches, denying learners opportunities for horizontal mathematization. These approaches are still dominant in most schools. In these approaches, leaners are first taught mathematical language (i.e. the terms to be used are defined at the onset) relevant to the piece of work to be done. Next, the teacher demonstrates an algorithm to solve problems related to the topic and, finally, learners are given an exercise or worksheet to complete (Barnes, 2004).

Transformational activities (which fall under vertical mathematization, as it mainly deals with manipulation of symbols) are over-emphasized in traditional teaching at the expense of representational, generalizing and justifying activities. A traditional approach forces most learners to memorize concepts, creating a strong possibility for forgetfulness and leaves them with no strategy in place to navigate a problem situation without an algorithm.

Maree et al. (2006, p. 230) added that, in South Africa, the teaching of mathematics both at school and university encourages “reticence, conformation to rules and it uses sophisticated language”, which students from poor learning environments do not have, and thus denying them the opportunity to develop their own strategies for solving non-routine mathematical problems. Mathematics teaching in less-advantaged schools is by far ‘drill and practice’, as opposed to initiating learning mathematicians into problem solving (Rand Study Panel, 2002).

Gravemeijer and Doorman (1999) noted that the traditional teaching of mathematics stems from what they call “a typical error” of instructional design by mathematicians or curriculum designers who tend to simplify a complex mathematical topic by breaking it up in smaller parts, which are then ordered in a sequence. They argued that this does not assist novices to see the parts as constituting the whole or how the pieces fit together to make a whole. This kind of teaching, which does not foster formation of relationships between concepts at the onset, is likely to inhibit future learning.

Kilpatrick et al. (2001) bridged the dichotomy between traditional and progressive teaching by describing five interwoven strands (competences) of mathematical proficiency, which a successful problem solver should demonstrate at the end of a learning process. When dealing with algebra, the strands of mathematical proficiency are subsumed into three algebraic activities (see section 3.2). However, Kilpatrick et al.'s (2001) categorization seems to be silent on linguistic competence, which the South African school curriculum deems key in learning mathematics and solving problems in financial mathematics. With this shortcoming aside, this researcher finds Kilpatrick et al.'s (2001) description of algebraic activities to be more encompassing and broad enough to capture the learning experiences of PSMTs in solving non-routine mathematical problems in financial mathematics, and thus the reason for its inclusion in the analytic framework for this study (see figure 3).

3.9 theoretical framework: Constructivist Problem-Solving Theory

The constructivist theory of learning arose from the work of Piaget. The premise for this theory is that logico-mathematical (conceptual) knowledge cannot be transferred from one person to another, but it is individually constructed (Murray et al., 1998). In support, Olivier (1989) argued that the mind of a problem solver is not a tabula rasa and knowledge cannot be transferred ready-made and intact from one person to another. In a similar manner, problem-solving strategies cannot be transferred ready-made and intact but they are constructed during mathematical problem solving by engaging in discussions, communication, reflection and negotiations. However, the construction of problem-solving strategies is solely a cognitive process, undertaken by an individual, irrespective of the stimulus. Mathematical concepts and problem-solving strategies are not concrete; hence, they do not exist in nature but exist only in thought.

A constructive perspective describes mathematical knowledge in general and problem solving knowledge in particular as an organization of facts, concepts, skills, problem-solving strategies and heuristics into units that form a web of knowledge called schema, which is activated during problem solving (Rodriguez et al., 2018). A cognitive constructive perspective enabled this researcher to account or explain the pervasive inconsistencies that are often found in the written work of learning mathematicians and possibly PSMTs. It acknowledges that PSMTs might have access to an appropriate problem-solving strategy (schema) but the stimulus (problem) may not cue the PSMTs into using the strategy. Hence, non-routine mathematical problem-solving means activating relevant schema and deactivating irrelevant schema in such a way that problem constraints are adequately used in the solution process. A focus on the cognitive aspect of problem solving is limiting as it seems to ignore the context in which the problem is found and the contributions thereof which this researcher deems critical in problem solving.

As soon as a strategy is constructed, it is validated and shared. From a cognitive perspective, problem solvers do not only interpret knowledge but they actively organize and structure knowledge into units of connected concepts and problem-solving strategies. Consequently, mathematical problem solving is entirely a cognitive process, whereby the problem solver intentionally and wilfully applies thinking in search of a solution in a social setting. Subscribing strictly to a cognitive perspective is found to be limiting, since social factors are not considered in knowledge production.

Problem-centred learning to teaching mathematics stems from the premise that individuals construct their mathematical knowledge and that this construction is a social activity (Murray et al., 1998). Mental processes exist between and among people in a social setting in which mathematical ideas are moved from the public space into one's cognitive realm. Mathematical knowledge consists of convention, symbols, notations, and rules which constitute linguistic knowledge and language is socially constructed (Department of Basic Education, 2011b; Murray et al., 1998). From a socio-cultural constructivist point of view, social interactions are therefore critical to problem solving in that they afford the problem solver an opportunity to make his or her thinking public. When individuals talk or write about what they know or how they know it, they reflect on it and examine what they say, and thus enabling more, or even new, connections to be made.

A problem creates a knowledge gap which needs to be bridged. Drawing on the work of Vygotsky, Goos (2004) argued that the construction of problem-solving strategies occurs within the zone of proximal development (ZPD). Crossing the ZPD may be an individual or a group endeavour. When PSMTs engage in a mathematical problem-solving activity, they are pulling each other forward into a mathematical inquiry. The problem solvers, in this case the PSMTs, continuously explore the problem situation to unravel the complexity of the problem (Kolovou et al., 2011). If on exploring the problem, the problem solver manages to construct efficient and effective problem-solving strategies, then the ZPD has been bridged, otherwise the problem solvers embark on seeking external assistance from peers, teaching mathematicians or other sources. The quest for a solution may compel problem solvers to change the strategy (intra-task or extra-task strategy) being used, as well as the algebraic activities, as more knowledge is gathered about the problem.

Furthermore, engaging in non-routine mathematical problem solving demands that problem solvers communicate their thoughts at each stage, and thus enabling them to regulate and monitor the solution process. In this study, pre-service mathematics teachers were required to write down their solution processes. This enabled the researcher to follow their thinking processes and determine the problem-solving strategies they used.

In this study, both RME theory and a constructivist theory on problem solving are used together because both concur that knowledge construction is both a cognitive and social endeavour. RME takes this point further by showing that problem solvers undergo horizontal and vertical mathematization during knowledge construction. However, RME, just like constructivism, does not clarify which mathematical tools are used by problem solvers in knowledge construction. The work of Kilpatrick et al. (2001) is useful here as it theorises algebraic activities which problem solvers use in knowledge construction. Kilpatrick et al.'s (2001) work does not seem to address contextualized mathematical problem solving in that it does not talk about the linguistic and contextual abilities that are needed to solve these kinds of problems, as indicated by the work of Pournara (2013) on financial mathematics.

Schleppegrell (2007) argued that language, in general, and mathematical language, in particular, impacts on problem solving in that the problem solver must be aware of both and be able to discern the mathematical relationships between the concepts. In the light of these concerns, this research has expanded upon Kilpatrick et al.'s (2001) work, by adding two new activities of algebra, namely linguistic activities and contextual activities.

Linguistic activities include activities such as writing convention (e.g. numbers are written before variables, i.e. $2x\ instead\ of\ x2$), ways of symbolizing (e.g. $y\ squared\ as\ y^2$. Contextual activities entail financial conventions such rand-based markets use 365 days in allocating interest, while USA markets use 360 days in allocating interest, understanding of financial convention and principles and ways of applying mathematical conventions, such as rounding. As indicated earlier, rounding in context goes against the usual conventions.

3.10 Theoretical framework: Espoused Problem-Solving Conceptions

Section 2.3 discussed that the inception of a democratic dispensation in 1994 in South Africa, was marked by the introduction of an outcome-based education system which was meant to unify the country's educational system and change classroom practices. The current South African curriculum advocates for teaching mathematics through problem solving, i.e. as an activity which mathematicians do. Problem solving is a progressive pedagogy associated with a socio-constructivist view in education (Handal, 2003). However, the teaching of mathematics in many South African classrooms still follows the traditional pedagogy associated with behaviourist education practices (Barnes, 2004). In these classrooms, learning mathematics is about mimicking the teacher.

ITE programmes are not exceptions when it comes to old traditions. (Adler, 2005) also contended that Advanced Certificate in Education (ACE) courses, aimed at teachers upgrading their qualifications in South Africa, were dominated by compressed mathematics (i.e. pure mathematics content), instead of unpacking (decompression).

Ernest (1989) and Holm and Kajander (2012) expounded that shifting classroom practices towards teaching mathematics through problem solving demands extensive changes in ITE programmes. Hiebert, Morris, and Glass (2003, p. 201) clarified the complexity of changing school practices by arguing that

> *"teaching is a cultural practice and changing cultural practices is notoriously difficult. People learn to teach, in part, by growing up in a culture – by serving as passive apprentices for 12 or more years when they themselves were students".*

Enculturation makes it easy for teachers to abandon new practices at the first encounter with a challenge (Mogari, 2014). For educational innovations to be of considerable value, Ernest (1989) argued that such changes should tap into teaching mathematicians' conceptions of the nature of mathematics, and how mathematics is taught and learned, as these impact on teaching mathematics through problem solving.

Holm and Kajander (2012) concurred with Bishaw (2011), that classroom activities are directly or indirectly influenced by the teachers' conceptions and their mathematical knowledge. Teachers' mathematical conceptions (what the teacher deems important and valid) influences how the teacher plans, organizes and executes classroom activities (Siswono et al., 2017). The mathematical knowledge of the teacher guides the enacting of these conceptions through the quality of questioning, feedback and directing classroom discussions (Siswono et al., 2017). Näveri et al. (2011) posited that conceptions are conscious beliefs which influence actions, while beliefs are an individual's subjective, experience-based knowledge about some matter. They argued that there are enacted and espoused conceptions. In this study, only espoused conceptions are investigated since the participants are studied while undergoing teacher training and not involved in teaching mathematics.

According to Ernest (1989), there are three hierarchical conceptions held by mathematics teachers about the nature of mathematics. Firstly, at the lowest level is instrumental conception, which is a view that mathematical knowledge is a collection rules, facts and algorithms with a utilitarian value to mankind.

Secondly, the Platonist conception views mathematics as objective, consistent and connected structure that exist external to human mind. Thirdly, a problem-solving conception, which is a view that mathematics is dynamic, located in social and cultural contexts. These conceptions of mathematics impact on the teaching and learning of mathematics as they influence lesson design, planning and execution.

In support, Spangenberg and Myburgh (2017) advanced that PSMTs' conceptions about the nature of mathematics do not only influence what they think about mathematics but also the ways in which they would teach and interact with learners. PSMTs' conceptions of mathematics are a brief summary of how their knowledge is structured, how they have come to know mathematics as learning mathematicians, and later as PSMTs, and how they are likely to teach mathematics once they are qualified as teaching mathematicians (Spangenberg & Myburgh, 2017).

This view is also shared by Schoenfeld (1985), who argued that an individual's mathematical view determines how one uses or does not use the knowledge at one's disposal. Elaborating this point, Ernest (1991, p. xiv), cited Thompson (1984), who argued that:

> *The observed consistency between the teachers' professed conceptions of mathematics and the way they typically presented the content strongly suggests that the teachers' views, beliefs and preferences about mathematics do influence their instructional practice.*

Conceptions about the nature of mathematics are not completely demarcated, and thus PSMTs may hold more than one conception at a given time, or they may profess one conception, while their practice aligns with another. In support, Holm and Kajander (2012) stated that conceptions depend on context and the classroom culture.

Zambo (1996) declared that espoused conceptions may not be enacted conceptions. He found that the espoused conceptions and practices of teaching mathematicians in his study did not align, since teaching mathematicians who spoke highly of calculators as being supporting tools in problem solving did not allow learning mathematicians to use them in problem solving activities. In support, Beswick (2005a) asserted that teaching mathematicians may hold beliefs which they do not profess and, in some instances, they may not be aware that they hold such beliefs. The connection between beliefs about the nature of mathematics, mathematics teaching and mathematics learning are shown in table 3. In this study, the classification of mathematical beliefs was deemed critical in describing PSMTs espoused conceptions of problem solving in financial mathematics.

Table 6: Beliefs about Mathematics, Teaching and Learning

Beliefs about the nature of mathematics	Beliefs about mathematics teaching	Beliefs about mathematics learning
Instrumentalist	• Content-focused with an emphasis on performance. • Teacher or textbook is an absolute authority. • The role of the mathematics teacher is to transmit clear information, demonstrate procedures for solving problems and explain the process of solving sample problems. • Tasks are broken down into smaller units that can be sequenced. • Learner errors are not tolerated. • There is no room for frustration and confusion.	• Skill mastery. • Passive reception of mathematical knowledge. • Practice learnt procedures by solving similar problems. • Learning means listening well to the teacher, and remembering everything the teacher told them.
Platonist	• Content-focused with an emphasis on understanding. • Teacher becomes a facilitator of knowledge.	• Active construction of knowledge.
Problem solving	• Learner-focused. • Mathematics is a set of relationships between concepts, procedures and facts. • Lesson starts with a challenging problem. • Frustration and confusion are understood to be a natural part of the learning process.	• Autonomous exploration of own interests. • Developing new methods for solving problems. • Learning results from struggling and making mistakes.

Source: Adapted from Beswick (2005a) and Siswono et al. (2017).

Ernest (1989) added that the disparity between teachers' conceptions and their classroom practices stems from the social context, i.e. expectations from fellow teachers, subject supervisors, the learning mathematicians themselves and institutionalized curriculum (such as the assessment policies/guidelines, learning and teaching support materials (LTSM) used and national schooling system). This study explored PSMTs' espoused conceptions of mathematical problem solving only, as these reflect the PSMTs' current thinking of problem solving. lecturers are interested in the decisions which teachers make regarding teaching and learning, so it is important to know what they think and know about teaching through problem solving as they exit ITE programmes.

Drawing on the work of Ernest (1989) and Makonye (2013) on teacher's conceptions of mathematics, this researcher has concluded that there are two competing conceptions on mathematical problem solving pedagogy, namely traditional and progressive problem solving conceptions. A traditional conception of problem solving is underpinned by the instrumentalist and Platonist view of mathematics. These views are frequently associated with the behaviourist ideology of teaching.

According to Handal (2003, p. 47), the traditional conception of problem solving:

> *Emphasizes transmission of knowledge and stresses the pedagogical value of formulas, procedures and drill, and products rather than processes… puts great value on isolated and independent learning, as well as conformity to established one-way methods and a predilection for pure and abstract mathematics.*

In contrast, there is a progressive conception of problem solving driven by a falliblist view of mathematics (Makonye, 2013). The progressive conception of problem-solving pedagogy is a social constructivism ideology, which acknowledges that the mind of a problem solver is not a tabula rasa and that context plays a major role in knowledge construction. In progressive problem solving, the problem solver is seen as an autonomous, active and capable participant in knowledge construction. Progressive problem solving gives recognition and value to new and informal problem-solving strategies which learning mathematicians bring into mathematics classrooms, by personally and socially constructing them as they ponder a solution to a non-routine mathematical problem. In this study, the exploration of PSMTs espoused conceptions was guided by the question: What are the espoused conceptions of PSMTs about problem solving?

3.11 Summary

In this chapter, the use of RME in guiding this study was propounded. RME was deemed appropriate for this study because it acknowledges that mathematical problems are solved by both formal and informal strategies. Under RME, novice problem solvers mathematize scenarios informally, and through progressive mathematization, they learn to mathematize problematic scenarios using mathematical symbolism, under the auspices of a knowledgeable other. In this way, learning mathematicians develop ownership of the reinvented mathematical ideas.

The reconstruction of mathematical ideas is supported by what Kilpatrick et al. (2001) termed algebraic activities. Engagement in algebraic activities requires problem solvers to use a variety of problem-solving strategies to check the validity and accuracy of their solution methods.

The next chapter provides an overview of the theories used in exploring the learning experiences of PSMTs in solving non-routine mathematical problems in financial mathematics.

CHAPTER FOUR: TOWARDS AN ANALYTICAL FRAMEWORK

4.1 Introduction

In the previous sections, the theoretical constructs (conceptions about mathematics, mathematization, heuristic, strategic and mathematical knowledge) that were used to explore the PSMTs' learning experiences in solving non-routine mathematical problems in financial mathematics were discussed. Figure 2 provided an overview of the analytical framework that was used in exploring PSMTs' learning experiences. The PSMTs' conceptions of mathematics have also been shown. It is this researcher's view that philosophical assumptions (implicit or explicit) about mathematics influence PSMTs' decisions about how a problem is solved. These assumptions cannot be directly observed from the written work of PSMTs. They can only be accessed through interviews and discussions.

Central to this framework are the components of mathematical knowledge, namely strategic knowledge, which consist of problem-solving heuristics and problem-solving strategies, disciplinary mathematical knowledge, linguistic mathematical knowledge, and the theoretical knowledge (mathematization). These constructs form the basis from which PSMTs' learning experiences were explored, identified and classified in relation to the research questions posed in section 1.11. This was achieved by analysing each written response, the audio-visual transcripts and interview transcripts, in line with the sub-constructs or analytical categories specified at the top of the framework.

4.2 Analytical Categories and Assumptions: Mathematization

As shown in the framework, mathematization is central to solving non-routine mathematical problems, since the problem solver (in this case the PSMTs) need to translate the verbal text into a mathematical representation or a model on which mathematical means may be applied. The mathematical means are embedded in algebraic activities. The application of algebraic activities would trigger the use of problem-solving strategies and heuristics. As Menon (2013) argued, for problem solvers to be successful in mathematization, they need to be able to:

- Identify assumptions necessary to solve the problem.
- Identify variables based on the assumptions made.
- Formulate models based on the assumptions and variables identified.

The demonstration of these competencies enabled this researcher to identify both the problem-solving strategies and algebraic activities in use. The framework further shows that the kind of mathematization in which one engages affects the algebraic activities on which one draws on in search for a solution, and the way the problem is solved, and vice versa.

4.3 Mathematical Knowledge

A problem solver's knowledge of mathematics (informal or formal) becomes a resource in solving non-routine mathematical problems. It enables the problem solver to select what is deemed essential in resolving an impasse. Heuristic knowledge (implicit or explicit) helps the problem solver in structuring and organizing the solution process. Strategic knowledge allows for a variation of strategies during problem solving, while algebraic activities provide the tools needed in arriving at a solution.

This researcher has acknowledged the partiality of this analytical framework. While it enables the researcher to explore pre-service mathematics teachers' learning experiences in line with the research questions, it obscures the complexity of the problem-solving process due to its sequential nature. For example, it gives the impression that when PSMTs are confronted with a non-routine mathematical problem, they need to know the heuristic to be deployed, a theory and the mathematical knowledge to be used. These would then lead them into using algebraic activities from which problem-solving strategies are born.

In practice, a problem solver does not need to know all these constructs to solve a problem, i.e. some of the constructs shown in the framework may be invisible to novice problem solvers, while they may be visible to seasoned problem solvers.

Problem solving begins at an individual level before it comes into the social context, and hence it is the personalized assumptions that influence a problem solver's actions. The social interactions needed in problem solving are not accommodated in the framework. As indicated earlier, problem solving is a messy process. It is both a social and cognitive activity; it is fluid, requiring problem solvers to move back and forth depending on the information gathered about the problem.

Although mathematical knowledge (problem-solving heuristics, problem-solving strategies, and algebraic activities) and mathematization (i.e. horizontal and vertical mathematizations) are distinct in the analytical framework, they may all come into play simultaneously during problem solving, and are influenced by the philosophical assumptions held by both individual PSMTs and the collective.

Interdependency and the connectedness of algebraic activities, heuristics, problem-solving strategies and mathematization, philosophical assumptions and the non-sequential nature of the problem-solving process are not explicitly captured in this framework. Further, the analytic framework does not show that problem solvers' perceptions about mathematics affect the way individuals mathematize (construct models/ representations of the problem) the problem. This tool is a partial representation of what happens in practice and is used here to examine research questions, and, hence, it is partial.

4.4 Problem-Solving Strategies

Engagement in mathematical problem solving demands flexible usage of problem-solving strategies and mathematical knowledge (Boaler, 1998). Krulik and Rudnick (1982), Posamentier and Krulik (2008) and Polya (2014) suggested that problem solvers may use intelligent guessing, solve analogous problem, counter example, amongst many other problem-solving strategies (see table 5).

Over the years, many countries have modified their school curricula to reflect the problem-solving skills and strategies which school leavers should acquire while at school. South Africa is not an exception. South Africa published a document titled, “Mathematics Teaching and Learning Framework for South Africa: Teaching Mathematics for Understanding” in 2018 as a means of conscientising and assisting teaching mathematicians in implementing CAPS through a problem-solving pedagogy (Department of Basic Education, 2018). The framework has advocated embracing and nurturing learning mathematicians’ intuitive strategies in learning mathematics.

The South African school curriculum stresses the importance of communicating mathematical ideas appropriately using verbal, graphical, numeric and algebraic modes. This researcher felt that the problem-solving strategies in table 5 may not, in their current form, adequately capture the learning experiences of PSMTs operating in a South African context. As a result, this researcher has condensed the problem-solving strategies in table 5 into (1) **verbal strategy** – which incorporates all the aspects of learning and using a language, (2) **algebraic strategy** – which entails symbolizing, writing algebraically (forming equations and expressions) and conventions used in algebra, (3) **pictorial strategy** – which entails all visual representations used in organizing the data to aid problem conceptualization (such as lists, tables, diagrams, charts, grouping, etc.), (4) **graphical strategy** – which entails using or sketching/drawing of graphs, and (5) **numerical strategy** – which involves the use of arithmetic. It should be noted that the use of educational technologies, such calculators, computers, cellular phones and so on, is implied in each of these strategies. For example, a problem solver may solve $x^2 + 7x + 6 = 0$ by simply choosing a quadratic function on a calculator and entering the required values. This process uses substitution to get the answers. Substitution is a critical component of algebra.

The first reason for this condensing is that some of the strategies in table 5, e.g. computing and intelligent guessing, often address the same mode of communicating mathematical ideas (numeric mode), formulating equations and expression and substitution mainly deal with algebraic mode of communication). Hence, this researcher felt it would be productive to formulate more encompassing problem-solving strategies.

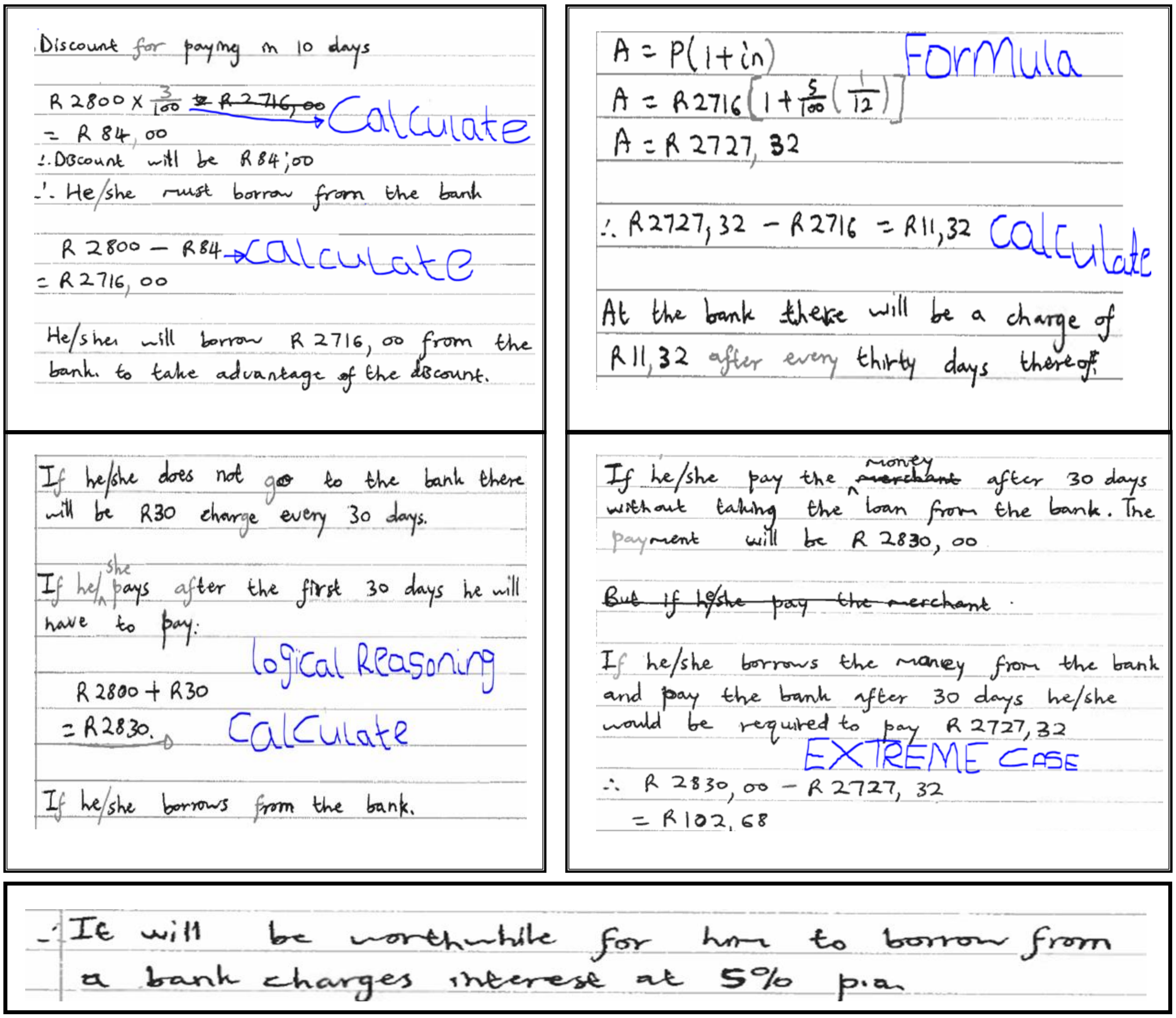
Secondly, in this study, some problem-solving strategies have appeared together in an intricate manner, as such that separating them would make the argument presented senseless. For example, calculating/computing and the extreme case, as used by the T_group, has indicated communicating mathematical understanding numerically (excerpt 2, row 2, column 2). Here, removing the calculations leaves the wording bare, and vice versa.

Thirdly, in this study, the problem-solving strategies suggested by Posamentier and Krulik (2008) seemed too specific and this researcher felt they could be used as codes, rather than categories.

Fourthly, some problem-solving strategies in table 5, when considered in line with the South African school curriculum for mathematics, seemed to lend themselves more readily to some learning outcomes more than others. For example, making a list, tabulating, and grouping, would easily be used when dealing with the learning outcome number and number relationships (Department of Basic Education, 2011a, 2011b, 2011c), whereas drawing a diagram would mostly be used when dealing with textual information.

However, one should be aware that whichever strategies are used is dependent on how the problem solver has conceptualized the problem, the knowledge resources which are called upon and the flexibility with which one uses the mathematical knowledge, the problem context and conditions. These five problem-solving strategies were used to explore PSMTs problem solving abilities in FM in relation to research question 1.

Excerpt 2: T_group Problem 1 Solution



4.5 Algebraic Activities

A number of researchers in mathematics education have pointed out that language is critical, not only in learning mathematics, but in solving non-routine mathematical problems, especially for second language speakers of the language chosen for teaching and learning (Bernardo, 2002; Englard, 2010; Sepeng, 2010, 2014; Setati et al., 2008). Problem solvers' struggles with the language impede their understanding of the problem and their conceptualization of the solution process.

Knowing that participants in this study studied mathematics in a second language and also informed by the work of (Pournara, 2013c, 2014, 2015b), as well as the NSC national diagnostic reports for the period 2011 – 2017, about learning mathematicians struggles in responding to FM questions, this researcher felt that Kilpatrick et al.'s algebraic activities were inadequate for this study. Consequently, linguistic activities were added to cater for issues relating to mathematical language and its conventions, while contextual activities were introduced to cater for issues relating to using mathematics in everyday life contexts. As a result, this study employed five algebraic activities, instead of three, in exploring PSMTs' learning experiences in solving non-routine mathematical problems in FM.

4.6 Summary

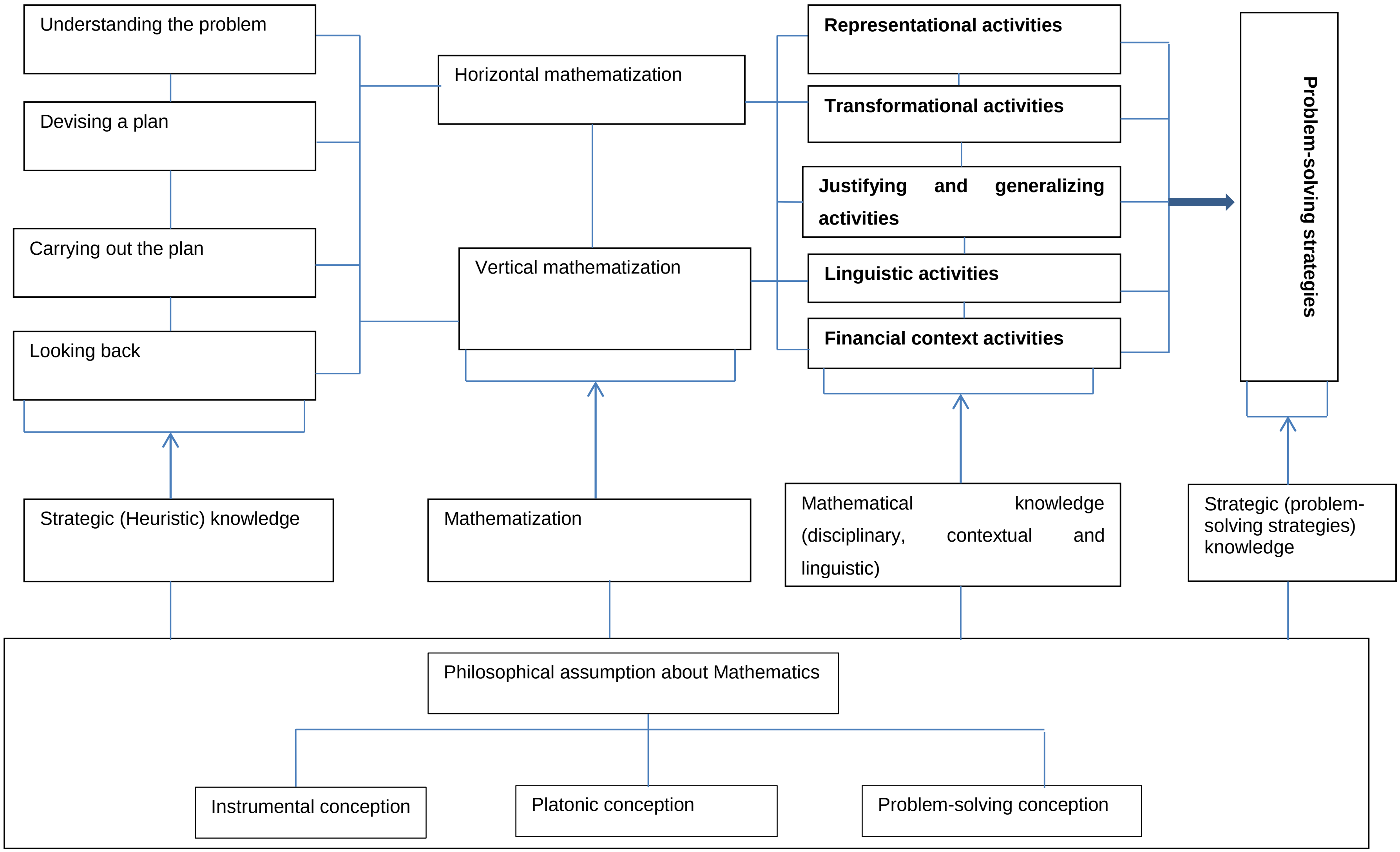
In this chapter, this researcher discussed the constructs that were used to explore PSMTs' learning experiences in solving non-routine mathematical problems in financial mathematics. These were the NCS aligned problem-solving strategies of algebraic strategies, graphical strategies, pictorial strategies, numerical strategies and verbal strategies. Kilpatrick et al.'s (2001) algebraic activities were also expanded upon by including linguistic and contextual algebraic activities. This study employed five algebraic activities in exploring PSMTs' learning experiences. These were contextual activities, linguistic activities, generalizing and justifying activities, representational activities and transformational activities.

Table 7: CAPS Aligned Problem-Solving Strategies

CAPS aligned problem-solving strategies	Problem-solving strategy	Description
Numerical	• Computing	• Includes straight application of arithmetic rules, order of operations and other procedures/algorithms. • May use tables and lists to organize the steps.
	• Intelligent guessing and testing (Trial-and-error)/ systematic guessing	• Making a reasonable guess, testing the guess and revising the guess if necessary. • Making lists or tables as a way of organizing the trials it needs a systematic approach.
	• Elimination of possible situations	• Identifying the end and the beginning of the problem.
	• Looking for a pattern	• Involves determining a common characteristic that can be generalized.
	• Solving a simpler equivalent case or reduction	• It demands maintaining the structure of the problem. • Rewording the problem. • Using smaller numbers. • Using a more familiar setting. • Dividing the problem into simpler problems.
	• Working backwards	• Reversing all operations.
	• Extreme case	• Involve calculating extreme values.
Algebraic	✓ Using a formula or writing equation or simplifying	✓ Entails substituting values into a formula, or expression, selecting the proper formula, formulating equations or algebraic manipulations,
	✓ Elimination of possible situations	✓ It involves eliminating possible solutions based on the information presented in the problem or elimination of incorrect answers.
	✓ Solving a simpler equivalent case or reduction	• It demands maintaining the structure of the problem. Includes: • Rewording the problem. • Using smaller numbers. • Using a more familiar setting. • Dividing the problem into simpler problems.

Pictorial	Organizing data • Making a table • Making a list • Making a chart • Grouping	Involves ordering data so that one can easily: • Identify and interpret patterns and relationships, • Select or pick the correct answer.
	• Drawing a diagram	• Using objects. • Using visual representations with all relevant information labelled.
	• Solving a simpler equivalent case or reduction	• It demands maintaining the structure of the problem. Includes: • Rewording the problem • Using smaller numbers • Using a more familiar setting • Dividing the problem into simpler problems
Graphical	• Analysing/drawing a graph	• Sketching/drawing graphs. • Drawing conclusions based on the graph.
Verbal	• Logical reasoning • Definition of terms • Reading aloud	• Making a series of inferences based on the information presented in the problem. • Mathematical operations are applied. • Requires reading between the lines.
	• Adopting a different point of view	• Requires one to find the most elegant way of solving the problem. This should be the method that is the less cumbersome and most efficient.

Figure 3: Summary of Analytic Constructs



CHAPTER FIVE: METHODOLOGY OF THE STUDY

5.1 Introduction

Chapter three discussed the literature on problem-solving, and particularly Poyla's (2014) four-step problem solving heuristic and the problem-solving strategies that are commonly used in mathematics. In chapter four, this researcher elaborated on the constructs that were used to explore PSMTs' learning experiences in solving non-routine mathematical problems and the theories that guided this study. This chapter outlines the research paradigm and design for this study, its rigour, limitations and methodology. The participants, context of the study, instruments, data generation and analysis process, and ethics are discussed.

5.2 Research Paradigm

A paradigm is defined as a basic set of beliefs that guides the actions of a researcher (Cresswell & Poth, 2018). This study has been guided by a constructivist philosophy. Constructivism, as a theory of learning and teaching, can be conceptualized from either a cognitive or social aspect of learning and teaching. Social constructivism, which arises from the work of Jerome Bruner and Lev Vygotsky, stems from the premise that knowledge is socially constructed (Liu & Ju, 2010). For mathematical knowledge, in general, and problem-solving strategies, in particular, to be of value, they must be validated and shared within the mathematics community. In this paradigm, participants have ample time for interaction, discussion and collaboration in the construction of a solution to the given problem (Cresswell & Poth, 2018).

The motivation for choosing a social constructivist paradigm is that it enables a researcher to fully describe the mathematical problem-solving strategies and algebraic activities used by PSMTs when confronted with non-routine mathematical problems, using qualitative methods of data collection and analysis.

In addition, Cresswell and Poth (2018) pointed out that this paradigm affords the researcher an opportunity to develop patterns of meanings inductively, by posing broad and open-ended questions that elicit what participants say or do in their life setting. From a cognitive constructivist point of view, the interest is in how PSMTs conceptualize and come to solve non-routine mathematical problems. This aspect of problem solving is abstract and beyond this researcher's observation capabilities. To illicit this, PSMTs were provided with non-routine problems to solve in a group.

Proponents of a cognitive constructivist paradigm believe that humans collectively, or individually, construct knowledge. The construction of problem-solving strategies is relative and dependent on the problem solvers existing knowledge and interpretation of the mathematical problem (Baxter & Jack, 2008). Researchers operating in this paradigm acknowledge the innovative and subjective creation of problem-solving strategies by a problem solver. The construction of problem-solving strategies involves both the cognitive and social contexts. Cognitively, the problem solver should be able to make mathematically sound connections between the concepts and operations identified in the problem in sophisticated ways, to devise a problem-solving strategy. Once such connections have been made, they are shared with others. Thus, problem-solving strategies are the final product of cognitive and social constructivism.

In this study, PSMTs collaborate in search of a solution to relevant non-routine mathematical problems in financial mathematics. A constructivist paradigm was chosen for this study because it allows a close collaboration between the researcher and the participants, enabling the researcher to observe PSMTs as they grappled with a mathematical problem. Through their discussion of the problem, the PSMTs described their views on how to solve the problem and what constitutes a valid problem-solving strategy. This enabled the researcher to not only better understand the PSMTs thinking, but also to document the algebraic activities which they used.

5.3 Research Design

This study aimed to explore the learning experiences of pre-service mathematics teachers with special reference to their non-routine financial mathematics' related problem-solving strategies. The study also aimed to explore the algebraic activities pre-service teachers contended with during such problem-solving exercises. The study employed a qualitative research design and the research approach was an exploratory single case study, with embedded units of analysis (Baxter & Jack, 2008).

Cresswell and Poth (2018) argued that, in qualitative research, researchers study subjects in their natural setting (i.e. where they live and work), to make sense of or interpret a phenomenon. In this study, an institution of higher learning provided the setting in which the problem-solving strategies of groups of pre-service teachers were explored to understand how PSMTs' learning experiences can be enhanced in the context of increasing their non-routine problem-solving competences.

Unlike quantitative studies, which rely on the collection of huge amounts of data, case studies give a detailed account of a case (Opie & Sikes, 2004). Kumar (2011) argued that a case could be an individual, a group, a community, an instance, an episode, an event, a subgroup of a population, a town or a city. Kumar (2011) further stated that case studies provide an in-depth understanding of the case, without claiming to make any generalisations to a population beyond cases similar to the one studied. In this study, a cohort of pre-service mathematics teachers, majoring in mathematics, in their final year, constituted the case. Each group into which the cohort was divided was considered to be a 'participant group' in the activities administered, as part of the research, and, hence, constituted the unit of analysis.

5.4 Rigour in Qualitative Studies

The rigour of research in qualitative inquiry is the extent to which other researchers can see the truthfulness of an inquiry. Unlike quantitative researchers, who ensure rigour by addressing reliability, validity, objectivity and generalizability of the research process, qualitative researchers address the trustworthiness of their research by ensuring that the research process is guided by four pillars, namely **credibility**, **transferability**, **confirmability** and **dependability**. These constructs are separately discussed in sections 5.5 – 5.8.

To reduce any possibility of anxiety due to the intrusiveness of the researcher, the study was conducted in the natural setting of all parties concerned and participants were asked to sign a declaration indicating that they were neither forced nor coerced to partake in the study. Furthermore, participants were informed that they may withdraw from the study at any point, without reporting or giving a reason to the researcher, and that the study would not form part of their learning activities. A direct consequence of this is that from the seventeen PSMTs who volunteered, only eight participants took part in solving the last two problems. Conducting the study outside the normal working hours of the institution ensured that the data were collected from those who were genuinely willing to participate.

Shenton (2004) posited that credibility is also enhanced by integrating various sources of data. In this study, data were generated through observations, interviews and documents (participants' responses). The researcher authenticated the transcripts prior to analysing them by listening to the audio recordings as he reads through the transcript. The researcher's claims are supported by excerpts from the transcripts as well as findings from similar studies. The participants, site of data collection and data analysis processes are sufficiently described (sections 5.11 – 5.16).

5.5 Credibility of a Qualitative Study

According to Ary, Jacobs, Irvine, and Walker (2018), credibility in qualitative inquiry is concerned with the researcher's ability to establish the truthfulness of the findings of the research processes, to give confidence to the audience about the findings of the inquiry, based on the research design, participants and context. The credibility of a qualitative inquiry may be enhanced by triangulation (structural corroboration), evidence based on consensus, member checking (evidence based on referential or interpretive adequacy), evidence based on theoretical adequacy, control of bias and the citation of legitimate sources (Ary et al., 2018; Creswell, 2002). Sections 5.5.1 – 5.5.5 investigate the elements of credibility and how they inform this study.

5.5.1 Triangulation (structural corroboration)

In this study, triangulation means the corroboration of evidence from different individuals (in this case, PSMTs in groups or as individuals); different sources or types of data, for example, sources of data in this study were documents (PSMTs written responses); audio and video recordings, observational notes and interviews recordings (Ary et al., 2018; Creswell, 2002; Tracy, 2019).

This researcher examined the documents and transcripts of each data source and formulated themes, which were used to describe problem-solving strategies and the algebraic activities used by the cohort of PSMTs in this study.

In support of this process, Ary et al. (2018) posited that triangulation would help a researcher to determine whether the data generated using one instrument may be confirmed by data generated using another instrument. This researcher examined the alignment of the written responses and the transcripts of the discussions that initiated the written responses. This helped in determining whether the submitted work was a group output or a solo effort.

5.5.2 Member checking (evidence-based on referential or interpretive adequacy)

Ary et al. (2018), in support of Creswell (2002), defined member checking (evidence based on referential or interpretive adequacy) as the process in which the researcher uses the participants to validated the accuracy of the transcripts, interpretations and conclusions made by the researcher. Member checking was critical in this study at the transcription stage, since some of the conversations happened through the home language (isiZulu) of the participants. While the researcher requested knowledgeable persons to transcribe and translate the conversation, it was necessary that the participants confirmed the fairness, representativeness, consistency, completeness and accuracy of the meaning of the translated version. To this end, this researcher requested the participants to avail themselves to scrutinize the transcripts and translations by listening and viewing the group's deliberations.

A full detail of the member-checking process has been provided under transcription and translation. Any claims or themes made by the researcher are supported by excerpts (verbatim or direct quotations) from the transcripts or written responses to show how PSMTs experienced mathematical problem-solving. This researcher used audio-visual recordings to verify the accuracy of the findings at a later stage by reviewing the records. As the study unfolded, this researcher continually applied external auditing by involving an individual knowledgeable in research but external to this researcher's field to give feedback on coherence of the study (Ary et al., 2018).

5.5.3 Evidence based on theoretical adequacy

Evidence based on theoretical adequacy relates to the extent to which a theoretical explanation developed from the study fits the data and is defensible (Ary et al., 2018). In this study, theoretical explanation was either deductive, in which case theorized problem-solving strategies and algebraic activities are clearly visible from the PSMTs' work or conversations, or inductive, in which case the researcher formulated problem-solving strategies and algebraic activities based on what PSMTs did and said as they solved non-routine mathematical problems. The excerpts from PSMTs written work or transcripts support inductively formulated strategies and algebraic activities.

Data generation took place over a one month period. This gave the researcher sufficient time to build a rapport with the participants. During this period, the researcher observed a range of activities which contributed to the data interpretation. Generating data over an extended period helped the researcher gain trust and possibly more honest responses from the PSMTs and also minimized participants' social bias of acting differently due to an intruder's presence or producing socially accepted answers.

This study used a number of theories, such as RME, constructivism (social and cognitive), and mathematical proficiency (Barnes, 2004; Kilpatrick et al., 2001), to explore PSMTs' problem-solving strategies.

5.5.4 Evidence based on control of bias

In qualitative research, the researcher is the main instrument for data generation. The researcher brings a wealth of experience which might block or extend his or her research horizon. Ary et al. (2018) seemed to share the same sentiments by arguing that bias "may result from selective observations, hearing only what one wants to hear, or allowing personal attitudes, preferences, and feelings to affect interpretation of data".

Based on research findings and personal observations, this researcher came into this study with an assumption that this cohort of PSMTs had a limited knowledge of problem-solving strategies. To minimize this researcher's bias in this study, he engaged in negative sampling (Ary et al., 2018) . Ary et al. (2018) defined negative sampling as an intentional and purposeful process in which the researcher seeks the opposite of what one expects. This researcher further sought to identify his own bias by reflecting on the research processes with the hope that knowing them (researcher's biases) would facilitate the documentation of interpretations and findings of this study.

5.5.5 Evidence based on consensus and citing legitimate sources

Evidence based on consensus is defined as an "agreement among competent others that the description, interpretation, evaluation, and thematics" are correct (Ary et al., 2018). The constructs developed and used in this study (e.g. linguistic and contextual activities, numerical, algebraic, verbal, pictorial and graphical problem-solving strategies) were derived from the existing literature on mathematics education and two South African educational policies, e.g. CAPS for mathematics (Department of Basic Education, 2011a) and "Mathematics Teaching and Learning Framework for South Africa: Teaching Mathematics for Understanding" (Department of Basic Education, 2018) and, thus, this researcher assumed that there is an agreement among competent researchers.

The sources used throughout this thesis are academic publications (books, journal article, conference proceedings), educational materials and government policies downloaded from sites ending in .edu, .ac or .gov. These sources were selected based on their authoritative information (Ary et al., 2018).

5.6 Transferability

The generalizability of findings is the distinctive feature of quantitative research. The defining trait for qualitative researchers is a transferability of findings. The transferability of findings is defined as the extent to which other researchers, working in similar circumstances, can draw, relate or apply the findings of a qualitative inquiry into their own context or the replication of research by other researchers, in a similar context or conditions. This definition distinguishes qualitative inquiry from quantitative inquiry. In qualitative research, other researchers (not the researcher) make generalization about the findings of the research. Tracy (2019) called this **naturalistic generalization** – a process in which other researchers value the findings of another inquiry and instinctively apply them in their situation.

In contrast, in quantitative inquiry, the researcher makes generalizations. The only requirement which a qualitative researcher has to meet is **descriptive adequacy,** i.e. the provision of accurate, rich, detailed and complete description of participants and context (Tracy, 2019). This would enable potential users of the findings to make informed decisions about the similarities and determine the extent of transferability (Ary et al., 2018; Tracy, 2019).

A transferability of findings is enhanced by conducting collective case studies, which allow for cross-case comparisons (Lune & Berg, 2016; Tracy, 2019). This study has used a single case study with embedded units (groups) (Baxter & Jack, 2008), which some researchers, such as Lune and Berg (2016), have named an instrumental case study instead. Using this type of case study allowed the researcher to make intra-group and inter-group analyses. Similar findings between groups corroborate the presence of a particular problem-solving strategy or algebraic activity, consequently maximizing the transferability of findings to other settings or contexts.

5.7 Dependability

Dependability refers to the extent to which the audience can depend on the findings of the research. In this study, excerpts are used to substantiate the themes developed, and this indicates that the findings are based on and originate from the data. This study uses several methods to generate data (documents, interviews and recordings). The data from PSMTs' written responses is available in appendix H.

A number of methods, such as audit trail, triangulation, code-recoding and replication logic, are applied to augment the dependability of the findings (Ary et al., 2018). The documentation through audit trails provide a mechanism by which others can determine how decisions were made and the uniqueness of the situation. An audit trail documents the research process and contains the raw data (see appendix H for PSMTs written responses) gathered in interviews and observations. Consequently, in this study, the data generation and data analysis processes were discussed in sections 5.15 and 5.16.

In this study, PSMTs solved non-routine mathematical problems in groups. Conducting the study using multiple groups increased the possibilities for dependability on the findings by the replication logic method.

Ary et al. (2018, p. 503) posited that according to this logic "*the more times a finding is found true with different sets of people or in different settings and time periods, the more confident the researcher can be in the conclusions*."

The code-recode and the corroboration of data sources enhanced dependability. During data analysis, this researcher coded the data from various data sources, per the instrument, over two sequential cycles. This means that there was a time lapse between the first coding and the second coding. The final themes were decided upon after comparing the themes from the first and second coding, within and across data sources.

5.8 Confirmability

Confirmability is the extent to which the findings of this study may be replicated by other researchers working in similar conditions and contexts. It also relates to the neutrality of a researcher in conducting the study and in reporting the findings. The researcher provided a detailed account of the participants and context so that others could verify the validity of the findings.

5.9 Limiting Constraints

This study used an instrumental case study with embedded units. Participants were drawn from one ITE programme in a rurally-based institution of higher learning with relatively low entrance requirements. Only eleven PSMTs participated in the problem-solving sessions hence the sample size was a limitation of this study. This implies that the findings of this study may not be extended beyond the participants. This cohort of PSMTs was selected based on the number of mathematics modules which they had completed since their first enrolment into the programme instead of the mathematics modules that they were doing. This could have influenced how they solved the problems. In addition to this, the participants' educational background (coming from less resourced schools) and the institutional location might have played a role in how these PSMTs solved non-routine mathematical problems in financial mathematics. The data generation process for this study took place between students' protests and end of year examinations. This could have affected how the participants responded and how the researcher conducted the interviews.

5.10 Research Design

This study used a case study methodology in exploring PSMTs learning experiences in solving non-routine mathematical problem. A case study design was deemed appropriate in exploring PSMTs problem-solving strategies because:

a) *One wants to answer a research question of type "what", "how" and "why" questions,*
b) *One is unable to manipulate the behaviour of the participants,*
c) *One wants to cover contextual conditions deemed critical to the phenomenon under study; or*
d) *The boundaries are not clear between the phenomenon and context.* (Baxter & Jack, 2008, p. 544)

Aligning oneself with Baxter and Jack (2008) views, this researcher employed a case study design in exploring PSMTs' problem-solving strategies. In this study, the unit of analysis was each group of PSMTs, while the case consisted of problem-solving strategies and algebraic activities applied to each non-routine mathematical problem in financial mathematics. Non-routine problems in financial mathematics provided the context in which problem-solving strategies and algebraic activities were developed and applied in resolving the problem. How PSMTs come to conceptualise and solve a non-routine mathematical problem is an activity which is beyond the researcher's ability and the way they solved the problem cannot be divorced from the context.

Woodside and Wilson (2003) argued that case study research is suitable if the objective of the research is to develop a deep understanding of PSMTs' problem-solving strategies by interpreting what they have done and how they resolved the problem. This researcher adapted four non-routine mathematical problems in financial mathematics to study how the cohort of PSMTs solved the problems. Polya (2014) opines that PSMTs apply heuristics when confronted with a mathematical problem.

In contrast, Kilpatrick et al. (2001) argued that non-routine mathematical problems are resolved by using algebraic activities.

A case study research design offered this researcher an opportunity to explore how PSMTs came to resolve non-routine mathematical problems. While there seems to be a consensus among researchers about the categories of case studies designs available, namely exploratory, explanatory and descriptive case studies, there is less clarity on the type of a case study which one may conduct (Ary et al., 2018; Creswel, 2009; Lune & Berg, 2016). For example, Lune and Berg (2016, p. 175) argued that case study researchers have to decide whether to conduct:

1) An intrinsic case study – wherein the case is chosen due to its uniqueness and when the researcher has vested interest in the case due to its peculiar features.

2) An instrumental case study – which is chosen "*to provide insight about an issue…the researcher focuses on a single issue or concern and identifies a single case to illustrate the item of concern….in these situations, the case actually becomes of secondary importance, playing a supportive role*" (Lune & Berg, p. 175).

3) A collective case study – which is composed of several instrumental case studies in different settings.

In contrast, Baxter and Jack (2008) posited that a researcher undertaking a case study approach must decide on whether to use a single, holistic or multiple case study. For this study, the researcher opted for a single case study with embedded units (Baxter & Jack, 2008), which is similar to an instrumental case study. This was adopted as this researcher believed that it would provide more information about the problem-solving strategies and algebraic activities used by this cohort of PSMTs in solving non-routine mathematical problems in financial mathematics. In addition, this type of case study was deemed appropriate because it allows inter and intra case analysis while foregrounding the issue under investigation (i.e. problem-solving strategies, algebraic activities and conceptions about mathematical problem solving).

Using a case study with embedded units enabled this researcher to analyse the case within a group, between the groups, and across the groups, and thus giving further insight into the problem-solving abilities of this cohort of PSMTs.

The case study methodology seems to be the most used method in studying problem-solving strategies and the strategy flexibility of both learning mathematicians and teaching mathematicians. For example, Mabilangan et al. (2012) investigated the problem-solving strategies of five participants (out of 124 third year students), in what they called ‘university high school’. Sepeng and Kunene (2015) studied Grade 6 learning mathematicians and Yew and Zamri (2013) examined the problem-solving strategies of pre-service secondary mathematics teachers (PSMTs) in Malaysia, enrolled in a 4-year Bachelor of Science with education (B.Sc. Ed.) degree. In all these studies, a single case study design was used with no intention to look at variation of problems-solving strategies within, between and across units. This study uses a single case study with embedded units which enables analysis within units, between units and across units.

5.11 Participants and Context of the Study

Participants in this study were drawn from a pre-service teachers’ bachelor of education programme at a South African university. The institution is one of other rurally-based universities in South Africa. As a result, most students enrolling in this institution come from less-resourced schools and mostly poor communities. Because of these factors, most of the students in this programme had the bare minimum of passing marks, such as a 40% in Grade 12 mathematics.

The participants in this study were in their final year of a 4-year bachelor of education programme, majoring in mathematics education. In this institution, final year students choose three major subjects per semester from a collection of eight modules (mathematics, physical science, life sciences, and technology). Hence, final year students have a choice of taking one, two or no mathematics per semester. The participants in this study had completed at least four semesters of “mathematics content” and two semesters of subject pedagogy “mathematics method”.

The sampling employed in this study was purposive sampling (Cohen et al., 2011), since it was based on the assumption that these pre-service teachers possessed various mathematical problem-solving strategies and could wrestle with algebraic activities during problem solving.

As Mills and Gay (2016) argued, the benefit of using a case study with purposive sampling is that information-rich informants are selected to enable the researcher to learn new things about the research problem. The researcher further assumes that their mathematical knowledge is fairly evenly matched across the groups and reasonably sound based on the number of mathematics modules they have passed since enrolling for the degree. In support, Cohen et al. (2011) posited that purposive sampling is used to access knowledgeable people.

In this study, it was also convenience sampling, since the students were registered and lectured to by the staff in the department to which this researcher belongs as a lecturer. This afforded this researcher easy access to the participants to conduct the study and less anxiety to the participants as they were familiar with the environment and this researcher.

A total of 26 invitations were distributed to PSMTs, of which 17 PSMTs agreed to participate and took part in responding to the open-ended questionnaire.

The second phase of data generation required students to solve four non-routine mathematical problems based on financial mathematics. In this phase 11 PSMTs came to the arranged session. The pre-service teachers were asked to group themselves in such a way that each group had a maximum of five members. Of the 11 PSMTs, it was discovered that two PSMTs were not taking financial mathematics module. Those two participants formed their own group. This arrangement ensured that their thoughts about the problems and the way they intend to solve them were not influenced by those taking the financial mathematics module. This gave another dimension to the study which this researcher had not planned on. This researcher did not seek to compare the strategies used by PSMTs taking a financial mathematics module to the strategies used by PSMTs who opted not to take a financial mathematics module.

Altogether there were three groups of PSMTs taking part in this study. The groups were given a non-routine mathematical problem in financial mathematics to solve each week for four weeks.

The participants were informed that all activities would take place outside of university working hours and that they agreed to convene at 16h30. They were further informed that their responses would be used to identify their problem-solving strategies and the mathematical knowledge they used. The problems were open to being solved in different ways, which were of varying levels of complexity, but all had one answer.

The non-routine mathematical problems were adapted from textbooks and research papers, for example, problem one (section 5.13.1) was adapted from (May, 1951), while problem two (section 5.13.2) was adopted from Pournara (2015),. The content coverage ranged from secondary school mathematics to Year 1 of university mathematics content. The problems were checked by experts for content validity and readability prior to being given to pre-service teachers.

Pre-service teachers were asked to solve the problems in more than one way and to explain what informed their solution process. The sequencing (from less cognitive demand tasks to high cognitive demand tasks) of the problems was critical in this study since, in the initial stages, participants were still trying to understand what was required of them and learning to work with each other. In support, Cresswell and Poth (2018) argued that, in order for the qualitative researcher to collect high quality data, the researcher needs to build a good rapport with the participants. To achieve this, this researcher gave them problems which demanded less thinking at the beginning and increased the cognitive demands gradually. While these problems were expected to be challenging, they were not overwhelmingly difficult.

5.12 Pre-Testing of Data Generation Instruments (Piloting Instruments)

The piloting of instruments requires administering the instruments to a similar group of participants, in similar conditions to those of the actual study (Kumar, 2011).

This process is intended to critically examine whether the participants' understanding of the questions and the meaning conveyed by the questions is similar to what the researcher intended. This researcher chose three PSMTs who were in their third year of studies for this purpose. They were chosen because of their good academic performance and had completed at least five mathematics modules in their programme, which was very similar to the actual participants. This pre-testing helped this researcher to find and eliminate ambiguous questions on the questionnaire and assess the wording of the mathematical problems.

Voice and video recorders were used in generating the data required in exploring the PSMTs' problem-solving strategies when solving non-routine mathematical problems in financial mathematics. Piloting helped this researcher in familiarizing himself with the technological gadgets and to check the quality of the audio and visuals. This researcher also developed an observation schedule and an interview guide to capture critical discussion moments and direct discussions during the interview. The data from these instruments were transcribed verbatim. The interviews were conducted with the groups based on their responses. The interviews were recorded and transcribed verbatim.

Recorders were used because they enabled this researcher to listen to the conversations and observe the interactions which ensued during problem solving at a later stage. Using recorders afforded this researcher an opportunity to focus on the critical issues emanating from the discussions without fear of losing operational and presentational data. Operational data refers to the spontaneous conversations of participants, while presentational data include the answers to the questions post by participants as they strive to determine a solution during a problem-solving process (Woodside & Wilson, 2003).

While the written work of PSMTs did allow this researcher to identify the strategies and algebraic activities, it only indicated the product of the PSMTs' deliberations. That could have omitted other strategies or algebraic activities employed during the discussions. Audio-visuals were very useful in alleviating that loss of data since the visuals recorded all aspects of the PSMTs problem-solving behaviour related to problem-solving strategies and algebraic activities (Opie & Sikes, 2004). The recorders proved to be useful in situations where PSMTs discussed the problems or explained their solution processes to each other without writing anything. It is during these informal discussions where some elegant problem-solving strategies were suggested, tested, validated or rejected, sometimes without having been written down.

Responding fully to the research questions on problem-solving strategies and algebraic activities demanded that this researcher observed the PSMTs in action. This demanded that this researcher made decisions on the order or sequence in which the various groups of PSMTs solved the four problems. This researcher decided that all the groups of PSMTs would solve the same problem, at the same time, but work from different venues. This was necessary to avoid a sharing of the problem-solving ideas across the groups. It was at this stage where this researcher became aware that observing all groups of PSMTs in situ was practically impossible, and hence an observation schedule alone would not be sufficient to respond to the research questions posed in section 1.11, since the researcher would not be able to be physically present all the sessions to be observed. To avoid losing or omitting some important data, i.e. verbal interaction amongst PSMTs (Opie & Sikes, 2004), voice and digital video recorders were deployed. The researcher felt that the use of technological devices would enable repeated observations at later stages and the validation of the interpretations of the transcripts.

As a result, an observation schedule, which the researcher thought would be the main data generation instrument, had to be a secondary data generation instrument. A digital video recorder was very helpful in this study since the interactions taking place as they unfolded, together with the non-verbal signals used, were recorded. A digital video recorder unlike a voice recorder records both voices and movements. Conducting parallel problem-solving sessions in different venues meant that this research could only observe one group per session and record the other two groups.

5.12.1 The questionnaire

This study used an open-ended questionnaire. The questionnaire was adapted from (Beswick, 2005a). It focused on conceptions about the nature of mathematics, conceptions about teaching mathematics and conceptions about learning mathematics. There were 26 items on the questionnaire. In piloting the questionnaire, three third year students in the same programme were used. Those students were like the participants, in relation to programme of study, home language and language of teaching and learning but they had not completed as many mathematics modules. These students did not have problems with the language used in the questionnaire.

5.12.2 Non-routine problems issued to PSMTs

The PSMTs worked on four problems adapted from textbooks and research articles. The researcher requested colleagues knowledgeable in financial mathematics to moderate the problems in advance. The merchant problem was based on simple interest and sought to access PSMTs' knowledge of the time value of money and interest. The other three problems were based upon the PSMTs' knowledge of compound interest and the time value of money by including loan repayments and annuities. All the problems required PSMTs' to make some assumptions and could be solved in many ways. Section 5.13 gives an overview of each problem.

5.13 An Overview of Non-Routine Problems Solved by PSMTs

5.13.1 Problem 1

This problem was based on simple interest, simple discount and unit conversion. These concepts are part of the South African school curriculum. At school level, this problem would be addressing the learning outcome number and number relationships stated in NCS – CAPS for mathematics (Department of Basic Education (2011a).

> *Problem 1*
> *A merchant receives an invoice worth R2800 payable in 30 days. The invoice states that if payment is made in 10 days, there will be a 3% p.a. simple discount. If the payment is not made in thirty days, there will be a R30 delinquency charge for every thirty days thereof. Would it be worthwhile for him to borrow from a bank that charges interest at 5% p.a. simple interest in order to take advantage of the discount? Justify your answer mathematically?*

Source: Adapted from May (1951).

This researcher had not expected the PSMTs to be confused by the problem as it drew on the everyday context of buying goods. The problem was chosen because this researcher believed that it could be solved using different approaches and strategies. The problem required the PSMTs to make assumptions and use them to solve the problem.

For example, the PSMTs needed to be explicit about how much money was to be borrowed from the bank and for how long. The PSMTs had to explicitly deal with the time value of money (TVM) principle and unit conversion.

5.13.2 Problem 2

This problem addressed compound interest and annuity concepts. Annuities are a critical component of FET mathematics curriculum in South Africa (Department of Basic Education, 2011a). This problem could be solved using different approaches and strategies. PSMTs were required to make explicit assumptions in solving this problem and deal with multiple cash flows.

> *Problem 2*
>
> *You open a NedTerm account with Nedbank in April 2008. This account allows you to make monthly deposit. When you open the account you make an initial deposit of at least R1000.00 and then monthly payments of R100 or more. You decided to deposit initial amount of R1000.00 and monthly deposit of R300. These payments are made at the start of every month and you do this for 18 months. The monthly payments start in May 2008. The interest rate is 7.65% p.a. compounded monthly. How much money will you accumulate over this period?*

Source: Adopted from Pournara (2015).

The wording of the problem seemed to have used simple language. The problem is situated within a banking context which could have been used as a resource by the PSMTs. However, solving the problem did not necessarily require the usage of the everyday banking practices.

5.13.3 Problem 3

To solve this problem, PSMTs needed to access the following financial concepts compound interest, loan repayments and deferred annuities and deal with multiple cash flows.

Problem3

A debt of R200 000 bearing interest at 5% p.a. compounded semiannually will be retired by R50 000 payment in 2 years plus a sequence of 8 equal semiannual payments starting at the end of 3rd year. Find the regular payment.

Source: Adapted from May (1951).

The problem made use of many technical terms. This could have led to the problem being misinterpreted by the PSMTs. This problem deviates slightly from the common practice of paying loans, where a lump sum is usually paid at the end to settle the debt.

5.13.4 Problem 4

This problem dealt with multiple cash flows and PSMTs needed to access knowledge of annuities (i.e. annuity certain) to resolve the problem. Annuities are part of the FET curriculum in South Africa.

Problem 4

A man, aged 25 pays R5000 at the end of year for the next 35 years to an investment company, he will receive R300 000 at the age of 60. At what annual rate is the interest being credited?

Source: Adapted from May (1951).

This problem differs from most problems given at school in that it requires PSMTs to calculate the interest rate i. At school, learning mathematicians are usually required to calculate either amount/value of investment F_v, the value of regular deposit/repayment x or the number of deposit/repayments n.

5.14 Data Collection Instruments

Data for this study were generated using interviews, observation, a questionnaire, and documents (Kumar, 2011). PSMTs solved problems in groups, sitting in different locations. Consequently, the researcher could not observe all the groups in person. The groups were either voice or video recorded as they solved the problems.

In support of the electronic recording of observations, Kumar (2011) posited that electronically recorded observations allows the researcher to review the interactions before passing any judgment, or interpreting interactions, and the researcher may request the views of other competent persons to make an objective conclusion about the interactions based on the recordings.

This researcher assumed a non-participant role during the observation. Taking this position, afforded this researcher to jot down critical conversations and observe the actions of the participants during the discussion. This study also sought to document the PSMTs' espoused conceptions of mathematical problem solving. This aspect was investigated using an open-ended questionnaire adapted from (Beswick, 2005a). The questionnaire consisted of 26 questions covering a range of conceptions about mathematics, mathematics teaching and the learning of mathematics. The questionnaire was self-administered to participants. This gave this researcher an opportunity to clarify the purpose of the study, its relevance and attend to any questions raised by the participants. The questionnaire was collectively administered (Kumar, 2011) to prevent the PSMTs from sharing their responses.

This study used focus group interviews with PSMTs. These interviews were intended to probe the PSMTs further on how they solved the problems. Sharing similar sentiments, Kumar (2011) contented that focus group interviews should be conducted to explore the perceptions, experiences and understanding of PSMTs' common experiences in solving non-routine mathematical problem in financial mathematics.

This researcher chose the groups to be interviewed based on their group responses and developed an interview guide (Kumar, 2011) so that the interview would stay focused. This process allowed this researcher to check the validity of his interpretations and conclusions against those of the participants.

At the end of each problem-solving session, this researcher collected all the written responses of the PSMTs. These documents were analysed (Appendix H). The documents used in this study have served as primary sources of data, since they were compiled in the presence of this researcher and solely for this inquiry. This is in contrast to Kumar (2011), who classified documents as secondary sources of data, irrespective of who compiled them, and for what purposes.

As indicated earlier, there were three groups of PSMTs that partook in the data generation. The groups solved the problems in different venues, at the same time. This demanded that this researcher got additional support to monitor and man the recorder in each venue. To this end, this researcher made use of three assistants. The research assistants also helped with the transcription and translation of the audio-visual recordings.

5.15 Data Transcription and Translation Process

The study began with the assumption that the PSMTs' discussions would be in English, which is the language of teaching and learning at the institution. This would have enabled this researcher to transcribe all recordings. As the study unfolded, it became clear that a transcriber and a translator would be needed. This researcher realized that it would be difficult to find a translator who had knowledge of mathematics as well. Ultimately, a specialist was not used in the process. There were several reasons for this decision.

Firstly, the criteria would have been a person who is knowledgeable in English, isiZulu and mathematics. A person meeting this criterion would have been difficult to find. The reality would have been that three people would have probably had to be sourced and hired to ensure thoroughness of the process. That would have tripled the cost of the study and there would have had been limited control over the process as these people would most probably be in different locations.

Secondly, this researcher wanted to be hands on the transcription and translation processes. This researcher believed that data transcription was the first step in data analysis. In some cases, not all the words or actions of participants were recorded, either due to a lack of audibility or because they were considered irrelevant to this study. Hence, this researcher has acknowledged that the data transcription and translation process might have become a selective process, where some ideas were lost but others were retained. That would have skewed the data analysis and interpretation.

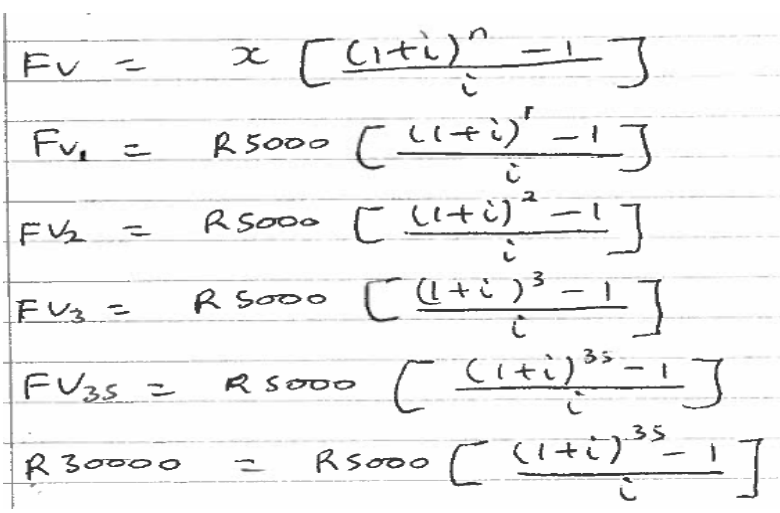
One reason for that is that the spoken mathematical language is not always the same as the written mathematical language. An example of this may be seen in the following excerpt 3:

Excerpt 3: T_group Problem 3 Transcript

Mkhize:	Masiqale, uthe it is difficult, iformula ithi F_v is equal to x into one plus i to the power n minus one all over i (Let us start, he said it is difficult, the formula **says Fv is equal to x into one plus i to the power n minus one all over i**).
Moon:	Sho (Sure).
Mkhize:	Sidinga u i, uthi ubani? Ifuture value yethu iyubani? iyu R300 000? (We want i, you said it is what? What is the value of the future value? Is it R300 000?).

The underlined words in excerpt 3 show how the transcription-translation process may interfere with mathematics. While the process depicts the verbal utterances, it fails to cater for the written mathematical aspects of the language used, as it does not refer to the brackets and division sign, as shown in the written response of PSMTs seen in the following response:

Excerpt 4 : T_group Problem 3 Solution



Because of this discrepancy, this researcher did not completely outsource the transcription-translation process. In this way, this researcher had an insider's perspective on both the PSMTs' problem-solving discussions by continually viewing and listening to the recordings while reading the transcripts.

This researcher also held regular meetings with the research assistants to discuss parts of the transcripts. The effects of transcriber and translator on the PSMTs' problem solving were kept at a minimum by having regular meetings and conversations with the research assistants. The transcription-translation process was undertaken in phases. Each phase has been described in detail here.

5.15.1 Phase 1 Transcription-translation

The researcher listened to all the recordings and transcribed one video and one audiotape recording (note that the research had minimal conversational knowledge of isiZulu). The purpose of this phase was not to get the transcription correct but to check if the researcher understood the gist of each group's discussion during problem solving and to familiarize himself with the context in which certain words were used.

One transcript was printed and given to a research assistant to read while watching the video. This was done to ascertain the extent of similarities or differences of the researcher's transcription and understanding of the PSMTs discussions. It also gave the research assistant an opportunity to learn how to transcribe.

5.15.2 Phase 2 Transcription-translation

The research assistant was asked to transcribe all the recordings and to translate them into English. This researcher then read the transcripts while listening to the recordings. This researcher noted on the transcripts the time intervals on the printed transcripts where he felt changes were necessary. Any changes that were made were mainly due to the paraphrasing of the participants' words during transcription.

5.15.3 Phase 3 Confirmation of transcription-translation by participants

This researcher printed all the transcripts from each group. This researcher then asked the participants to reserve a time to verify the transcription and the translation by reading the transcripts, while listening to the recordings. The participants could make changes to the transcripts or suggest which section(s) should be modified. Not all the participants attended that activity. However, all the groups were represented. Only one group returned the transcripts without changes and suggestions. The participants suggested that there was a need to edit the isiZulu parts. This worried this researcher and warranted finding out whether the research assistant ever studied isiZulu. This research found that while the research assistant could speak isiZulu fluently, it was not his home language. It is at this point that this researcher decided to use the other two assistants in the transcription-translation process. The second assistant was in her final year of a Bachelor of Education degree specialising in Mathematics, Science and Technology Education and her home language was isiZulu. The third assistant was a PhD student in the Faculty of Science at the institution and had isiZulu as a home language.

5.15.4 Phase 4 Transcription-translation

Based on the participants' recommendations for the isiZulu to be reviewed, this researcher called in two assistants. The first assistant was knowledgeable in mathematics, isiZulu and English and was tasked with refining the transcripts by reading the transcripts, while listening to each recording. The second assistant was a postgraduate student at the institution knowledgeable in English and isiZulu and only worked on the electronic transcripts to check the consistency in the meaning across the two languages.

5.15.5 Written responses from PSMTs

After each problem-solving activity, the PSMTs' written solutions were collected, and any identifying information was removed. The researcher gave each group a name (i.e. M_group, D_group and T_group) to make sure that the work was categorised correctly. Similarly, the audio and video recordings were moved from the recording devices daily into password protected folders on this researcher's personal computer. The naming was critical because the interviews were based on each group's written work.

In preparation for data analysis, this researcher scanned all the documents written by the PSMTs during the problem-solving sessions and their questionnaire responses. All the data were stored in an electronic format, and per group.

5.16 Data Analysis

The data generated in this study were analysed based on the theoretical constructs developed in chapter 3 and chapter 4 (see section 3.3, 3.5 and 4.4). The data analysis required that this researcher made decisions on what was to be analysed and how it would be analysed.

The content analysis was used to identify the problem-solving strategies and algebraic activities used by pre-service teachers. Elo and Kyngäs (2008) argued that the purpose of using content analysis is to provide a thick description of a phenomenon. According to Graneheim and Lundman (2004), analysis could be focused on manifest content or latent content. They defined manifest content analysis to be the analysis of what the texts says, dealing with the content aspect, and describing the visible and obvious components, while latent content analysis deals with what the text talks about and deals with the implied aspects. In this study, manifest content analysis was used.

This researcher used an inductive method of content analysis to build categories that were used to describe the problem-solving strategies used by pre-service teachers in solving non-routine financial mathematical problems. In an inductive content analysis process, categories are generated from the data (moving from the specific to the general). For example, the italicized part of the T_group solution process (excerpt 5) indicated that the group started solving the problem by recalling the formula first. They then proceeded to identify the known and unknowns. This part was coded as using a formula and substitution. Recalling formula and substituting the values in it signify the use of algebraic language.

This means that PSMTs are employing **algebraic strategies and algebraic activities** in their solution process.

Excerpt 5: T_group Problem 4 Transcript

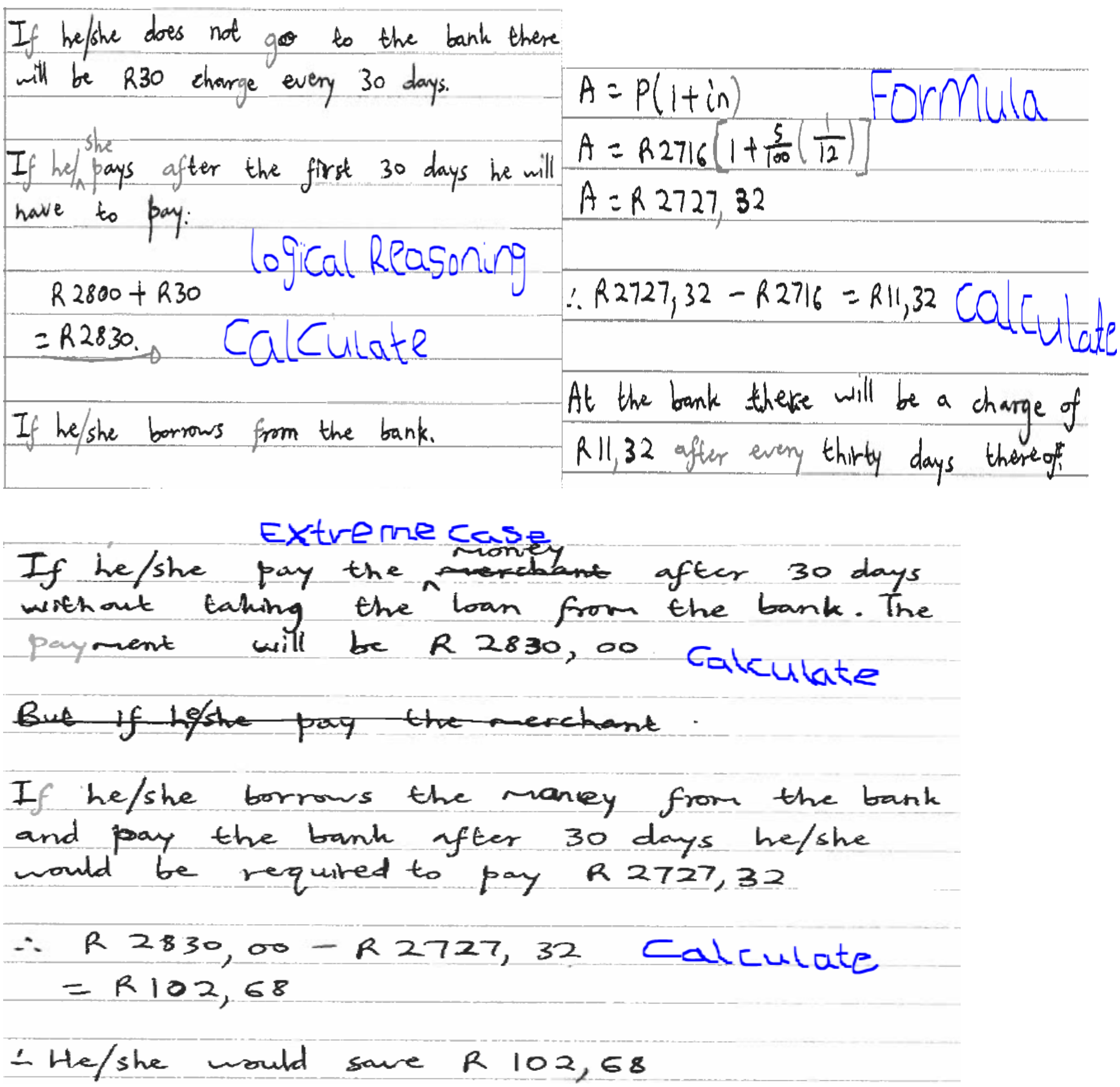
A:	Masiqale, uthe it is difficult, iformula ithi F_v is equal to x into one plus i to the power n minus one all over i (Let us start, he said it is difficult, *the formula says* Fv *is equal to* x *into one plus* i *to the power* n *minus one all over* i).	*Formula*
B:	Sho (Sure).	
A:	**Sidinga u i, uthi ubani? Ifuture value yethu iyubani?, iyu R300 000**? (We want i, you said it is what? *How much is our future value*? Is it R300 000?).	*Identifying unknowns*
B:	U R300 000, u x iwu R5000 (It is R300 000, x is R5 000).	*Substitution*

Inductive data analysis was applied on all transcripts, while deductive data analysis was applied on the written responses of PSMTs. Engaging inductive data analysis demanded that this researcher read the transcripts critically, coding each sections of the response which indicated the use of a strategy. This researcher divided the data into parts as he read it to authenticate the formulated code or heading. This researcher then read the codes and, where possible, generated more encompassing categories or themes. For example, the strategies formula, identifying unknowns and substitution were amalgamated into a more encompassing code (theme) algebraic strategy. The broad themes were developed and were used to describe the learning experiences of PSMTs in solving non-routine mathematical problems in FM.

Graneheim and Lundman (2004) contented that the formulated categories should be exhaustive and mutually exclusive, so that no data would be left out due to the unsuitability of categories and that no data should fall between the categories. While this requirement was satisfied with the broad themes, this researcher noted that some codes such as 'solving a simpler case' could be carried out within each broad theme.

In addition to analysing pre-service mathematics teachers' verbatim transcriptions of audio-visuals recordings, their written responses were analysed deductively to augment the inductive analysis. For example, in responding to the merchant problem the T_group wrote:

Excerpt 6: T_group Problem 1 Solution


If he/she does not go to the bank there will be R30 charge every 30 days.

If he/she pays after the first 30 days he will have to pay:

Logical Reasoning

R 2800 + R30
= R2830.

Calculate

If he/she borrows from the bank.

$A = P(1+in)$ Formula

$A = R2716\left[1+\frac{5}{100}\left(\frac{1}{12}\right)\right]$

$A = R\,2727,32$

∴ R2727,32 − R2716 = R11,32 Calculate

At the bank there will be a charge of R11,32 after every thirty days thereof.

Extreme case

If he/she pay the money after 30 days without taking the loan from the bank. The payment will be R 2830,00 Calculate

~~But if he/she pay the merchant~~.

If he/she borrows the money from the bank and pay the bank after 30 days he/she would be required to pay R 2727,32

∴ R 2830,00 − R 2727,32 Calculate
= R102,68

∴ He/she would save R 102,68

This group used various strategies, coded as compute/calculate, formula, logical reasoning and extreme case values, in solving the problem. On further analysis, these strategies were condensed into broader strategies, such as **verbal strategies** (e.g. logical reasoning), **algebraic strategies** (e.g. formula), **numerical strategies** (compute/calculate, extreme case values).

5.17 Analysis of PSMTs' Written Responses

Data for this thesis were generated using various instruments and methods. Concerning the analysis, this researcher had to decide whether to analyse the data per research question or per the method of data generation. The latter was chosen.

In analysing the written responses, the descriptions of the constructs (see chapter 4) formed the codes used in exploring PSMTs problem solving abilities. These codes were amalgamated to form categories, which are shown in the summary of analytic constructs (see figure 3).

This researcher read the responses of each group several times and indicated on the response the part which signified a particular code (see excerpt 6). For example, the T_group response to problem 1 (see excerpt 6) was coded for *calculate/compute*, *formula*, *extreme case* and *logical reasoning*. These codes represented NCS-aligned problem-solving strategies. The **numerical** problem-solving strategies were represented by calculate/compute and extreme case. Logical reasoning and definition of mathematical terms indicated the use of **verbal** problem-solving strategies, while recalling a formula and substituting values in it reflected the use of **algebraic** problem-solving strategies.

The PSMTs' ability to recall a formula and substitute values of each unknown in it, exemplified that PSMTs were working with **vertical mathematization.** Alternatively, arriving at a conclusion through logical reasoning was indicative of the PSMTs' engagement in **horizontal mathematization**.

In solving problem 1, the T_group (see excerpt 6) engaged with computations, substitutions, and conversion of percentages to fractions, which showed that they demonstrated the **transformational activities of algebra**. The T_group assumed that the client would be borrowing the money from the bank for 30 days to take advantage of the discount offer, which indicated their ability to make inferences based on the text, and thus signified an ability to engage in the **representational activities of algebra**.

As can be seen from the problem-solving processes of the T_group (see excerpt 5), their strong mathematical knowledge was also to their disadvantage, in that by assuming a 30 day (instead of 20 days left after the expiry of the discount period) lending period, they enriched the bank by taking more money from the client. Hence, this group committed a **contextual mathematical error** in their search for a solution. The failure of this group in resolving problem 1 was instigated by the time value of money (TVM) principle, which appeared to be a missing link in this group's mathematical knowledge.

Table 8 shows the overall quantified problem-solving processes across the problems for each group. To quantify the data, this researcher counted the number of codes that were amalgamated in that problem to form a category. For example, the solution of the T_group to problem 1, the category **numerical problem solving strategy**, was made from the codes calculate which has a count of four (4), extreme case with a count of one (1), and hence numerical strategy has a count of five (5), i.e. $4 + 1 = 5$, while **algebraic problem solving strategy** has a count of one (1) since it has emanated from a single code formula with a count of one (1). Similarly, **verbal problem-solving strategy** has a count of one (1), as it is based on logical reasoning with a count of one (1).

In table 8, reading horizontally across the table shows the strategies used by each group; the algebraic activities with which the group grappled during the problem solving; the kind of mathematization deployed in the solution process; and the type of errors made by the group in solving the problem. This reflects the extent of strategy variations within the group.

In table 8, reading down a column indicates the consistency in using a strategy; algebraic activity; and the kind of mathematization or type of error committed across problems and groups. A count of zero (0) indicates that the construct was not identified by this researcher.

Table 8: Quantified Composite Analysis of PSMTs' Written Responses

Problem	Group	Problem-solving strategy					Algebraic activity					Mathematization		Problem-solving approach		Types of mathematical errors		
		Numerical	Algebraic	Verbal	Pictorial	Graphical	Representational	Transformational	Justifying & generalising	Contextual	Linguistic	Horizontal	Vertical	Account	Individual	Mathematical	Representational	Contextual
1	T_group	5	1	1	0	0	2	7	0	1	1	1	5	0	0	0	0	1
	M_group	3	1	1	0	0	1	4	0	1	2	2	4	0	0	1	0	1
	D_group	2	1	0	0	0	1	4	0	1	0	0	3	0	0	1	0	1
2	T_group	3	4	0	0	0	2	7	0	0	1	1	6	6	0	0	0	0
	M_group	1	4	0	0	0	2	5	0	0	1	0	5	0	2	0	0	0
	D_group	3	2	0	2	0	4	5	0	4	0	0	8	6	2	5	1	3
3	T_group	4	6	1	0	0	2	6	0	1	0	3	6	4	1	1	0	0
	M_group	1	6	0	0	0	4	8	0	0	0	2	6	4	1	1	0	0
	D_group	3	0	1	1	0	1	4	0	0	2	2	3	3	0	2	0	1
4	T_group	0	7	0	0	0	2	5	0	0	0	0	6	0	2	1	0	0
	M_group	2	7	0	0	0	3	6	0	0	0	2	7	2	1	2	0	0
	D_group	3	2	2	0	0	1	3	0	2	1	2	2	3	0	0	0	0
TOTAL		30	41	6	3	0	25	64	0	10	8	15	61	28	9	14	1	7

5.18 Analysis of PSMTs' Transcripts

Prior to the analysis of the transcripts, this researcher read the transcripts several times to make sense of what was discussed. In some instances, this researcher had to watch the video or listen to the audio as he read the transcripts. It was during this process that this researcher decided on merging the transcripts with the written responses of the PSMTs. This seemed to offer insights into what was being discussed. For example, the members of the D_group talked about using a number line to solve problem 2, which did not make sense to this researcher until the written responses were merged with the transcripts. Merging the documents further helped the researcher in validating the accuracy of the transcripts. However, it may be noted that not all the discussed solution processes were written down.

The transcripts, just like the written responses, were analysed per problem across the groups, i.e. problem 1 was analysed for all the groups before proceeding to problem 2. In analysing the transcripts, the constructs in section 3.2 and 3.5 were used, while at the same time, this researcher was mindful of emerging constructs. For example, on reading the D_group's work (see excerpt 7), the phrase "number line" came to the fore and this resulted in a descriptive code timeline. The timeline indicated the use of the **pictorial strategy** and **representational activities** of algebra, since this group visually represented problem 2 as they endeavoured to find a solution to the problem. The code time line was used due to its relevance and dominance within South African LTSM. In this study, the timeline seemed to bring together both problem-solving strategies and algebraic activities. In mathematics, particularly in the early grades, a number line is used to evaluate numerical problems (i.e. to teach addition and subtraction).

Hence, a number line has been used as a tool and a strategy in solving the problem (see excerpt 7).

Excerpt 7: D_group Problem 2 Transcript

⋮

Mosotho: You can't do it with a number line.

Mthembu: I can but I want to say, oooh okay now I see, we say when, here is R1000 plus R300 lo awukhokhile for now is… (I can but I want to say, Oh okay now I see, we say when, here R1 000 plus R300 that he paid for now is…).

⋮

Mthembu: Uyobona nje first month up until 18 or else sezosebenzisa inumber line (you see the first month up to 18 or we will use the number line).

⋮

By using a number line, the D_group were able to horizontally mathematize problem 2 (i.e. move from the use of words to the use of symbols). Using a number line enabled the group to devise a plan to solve the problem. In this way, the group engaged in **representational algebraic activities** and **pictorial problem-solving strategies**.

The analysis of the transcripts proceeded in the same manner as the written responses, in which case, this researcher labelled the parts of the transcripts with pre-formulated constructs that matched the activities or utterances of the PSMTs. For example, in solving problem 3, the T_group engaged in a discussion, as seen in excerpt 8:

Excerpt 8: T_group Problem 3 Transcript

⋮

Mkhize: A man aged 25 pays R5 000 at the end of the year to an investment company, he will receive R300 000 at the age of 60. At what annual rate is the interest being credited?

Moon: Ok, uneminyaka ewu 25 njengemanje (A interjects) (Ok, he is 25 years right now).

Mkhize: Ya, uzokhokha ku investment company for 35 years (Yes, he will pay to the invested company for 35 years).

Moon: Mmmm, u R5000 (Mmmh, R5 000).

Mkhize: Ya, which makes 60 years wakhe, lo $25 + 35$ makes lo 60 years wakhe. At the age of 60 he will receive R300 000 (Yah, which makes 60 years, this 25 + 35 makes his 60 years. At the age of 60 he will receive R300 000).

Compute

Moon: Mmmm, lona shuthi ubani? (Mkhize interjects) Monthly payment? (Mmmh, meaning this is?) (Mkhize interjects) monthly payment?).

Mkhize: Ifuture value…. This is the x, we need u i kuthi ubani? (it is future value.... This is the x, we want to know what is the value of i).

Symbolising

Identifying value

Moon: Ok, ilula uma injalo mfethu (Ok, it is easy if it's like that).

Mkhize: It is simple.

Moon: Why athi it is difficult nje? (Why did he say it is difficult?).

Mkhize: Masiqale, uthe it is difficult, iformula ithi F_v is equal to x into one plus i to the power n minus one all over i (Let us start, he said it is difficult, the formula says Fv is equal to x into one plus i to the power n minus one all over i).

$$Fv = x\left[\frac{(1+i)^n - 1}{i}\right]$$

Formula

Moon: Sho (Sure).

Mkhize: Sidinga u i, uthi ubani? Ifuture value yethu iyubani? Wyu R300 000? (We want i, you said it is what? What is the value of the future value? Is it R300 000?).

Identifying unknowns and knowns

Moon: U R300 000, u x iwu R5000 (The R300 000, x is R5 000).

Mkhize: Ya R5000 (Yes R5 000).

Moon: For the next 35 years.

Substitution

Mkhize: Akesithi then sino one plus i, akesithi vele ukhokha yearly shothi asikho isidingo soku divider i (Let's say we have one plus i, so he pays yearly and there is no need to divide i).

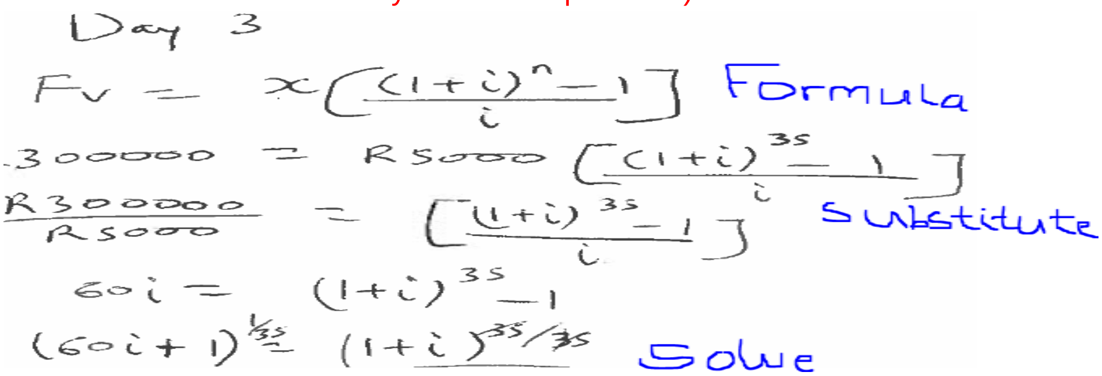
Moon: Sho, it is 35, nuh? (Sure, it is 35, nuh?).

Mkhize: Akesithi then sino one plus i, akesithi vele ukhokha yearly shothi asikho isidingo soku divider i (Let's say we have one plus i, so he pays yearly and there is no need to divide i).

Moon: Sho, it is 35, nuh? (Sure, it is 35, nuh?).

Mkhize: Then n is 35 years, minus one all over i. → **Substitute**

Moon: Then okay, asithi R300 000 divide by R5000 is equal to 60 (Then okay, let us say R300 000 divide by R5000 is equal to 60).



Mkhize: 60 is equal to…

Moon: U $i$plus one, 35 minus one

Mkhize: Then we have to multiply by i both sides

Moon: Ya (Yah).

Mkhize: Then sosala no one plus i power 35 minus one (The we will be left with one plus i power 35 minus one).

Moon: Lo one asimphushele ngapha azoba positive (Let's push one this side so that it becomes positive).

Mkhize: Then 60.

⋮

Excerpt 8 shows that the T_group used two problem-solving strategies to solve the problem, namely **numerical problem-solving strategy** indicated by the code compute and **algebraic problem-solving strategy** indicated by the use of a formula and *s*ubstitution. This researcher coded all the transcripts in a similar manner.

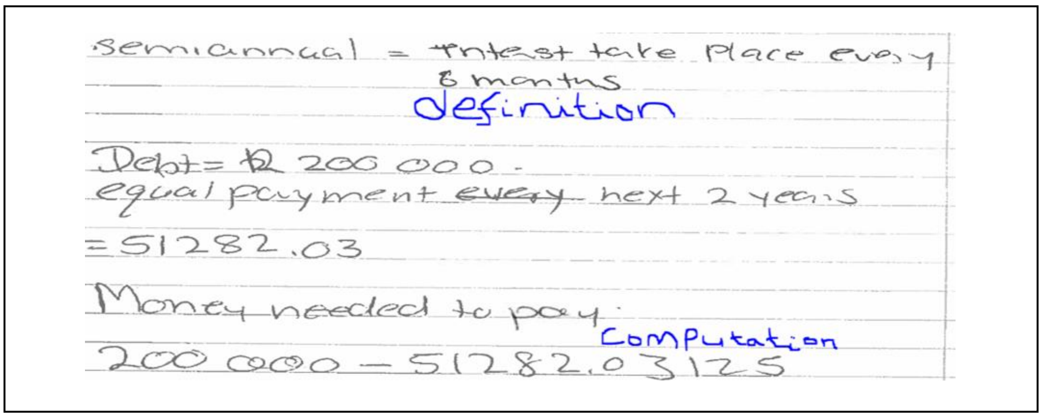
The overall quantified analysis is shown in table 9. It is important to note that the numbers refer to the number of instances that were coded as reflecting the construct.

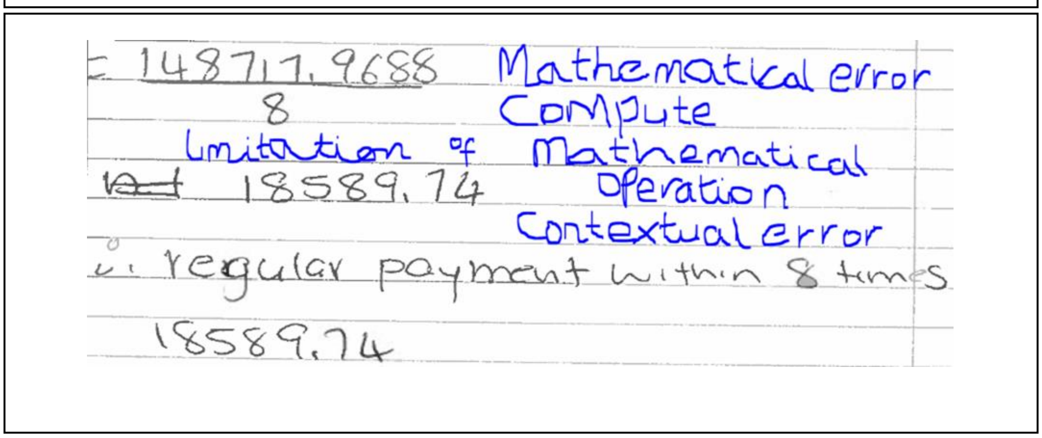
Table 9: Quantified Composite Analysis of PSMTs' Transcripts

Problem	**Group**	**Problem-solving strategy**					**Algebraic activity**					**Mathematization**		**Problem-solving approach**		**Types of mathematical errors**		
		Numerical	Algebraic	Verbal	Pictorial	Graphical	Representational	Transformational	Justifying & generalising	Contextual	Linguistic	Horizontal	Vertical	Account	Individual	Mathematical	Representational	Contextual
1	T_group	8	10	1			1	5			1	5	4					
	M_group	2	2					2					1					
	D_group	1	2	1							1	1						1
2	T_group	6	8	1				4			3		3	4	2	1		
	M_group		2					1		1			1					
	D_group	4	5	3	1		3	3			4			1		1	3	4
3 & 4	T_group	5	10		1		2	3					5	1		1		
	M_group	5	10	3			2	4			3		4	1	1	2		
	D_group	1		4	2						4							
TOTAL		32	49	13	4		8	22		1	16	7	18	7	3	5	3	5

To keep track of all the codes and themes (super codes), this researcher uploaded the transcripts on Atlas.ti 8.4.18 Trial version. Using this software enabled this research to see which codes were amalgamated into a particular theme. It also revealed which codes were infrequently used. The infrequently used codes appeared to be codes that described other aspects of the PSMTs' problem-solving abilities. For example, the code *time value of money* was used to show that the PSMTs' calculations ignored financial principles, while adhering to mathematical principles. For example, in determining the regular payment in problem 4, the D_group (see excerpt 9) divided the account balance by the number of payments, which ignored interest accrual:

Excerpt 9: D_group Problem 4 Solution


semiannual = intrest take place every 6 months
definition
Debt= R 200 000-
equal payment ~~every~~ next 2 years
=51282.03
Money needed to pay:
Computation
200 000 – 51282.03125


= 148717.9688 / 8
Mathematical error
Compute
Limitation of Mathematical operation
~~R~~ 18589.74
Contextual error
∴ regular payment within 8 times
18589.74

5.19 Analysis of PSMTs' Responses to the Open-Ended Questionnaire

The questionnaire required the PSMTs to provide a justification for agreeing or disagreeing with the given statements. Prior to responding to the questionnaire, the PSMTs were given numeric identifiers (instead of using their names) which were written at the top of the front page of each questionnaire. These numeric identifiers were used in classifying the PSMTs' responses (see table 10). This means that the numbers appearing under the column labelled 'PSMTs' responses' are not the frequencies but the numeric identifiers of the PSMTs. For example, in responding to item 17 on the questionnaire, which dealt with an absolutist's conception of mathematics teaching, PSMT 4 wrote:

> It is the teacher's responsibility to provide children with clear and concise solution methods for mathematical problems.
>
> I Disagree learner Should discover methody own their own with the guidance of a teacher.

The response of PSMT 4 favours a teaching approach that gives control and authority to the learning mathematicians to reinvent mathematical ideas and problem-solving strategies under the auspices of a knowledgeable other. According to PSMT 4, the teacher should assume a facilitative role in a teaching and learning situation. PSMT 4 seems to subscribe to a progressive approach in mathematics teaching.

In contrast, responding to the same item, PSMT 7 said:

> 17 It is the teacher's responsibility to provide children with clear and concise solution methods for mathematical problems.
>
> I Agree
> Teachers should assist Learners when they are Stuck correct them, Show them the correct ~~Step of~~ methods of Solving Maths.

PSMT 7's explanation places the teacher in an authoritative position in the foreground of the learning process. According to PSMT 7, the teaching mathematician has a repertoire of the "correct methods of solving maths" which should be transmitted to the learning mathematicians. This view aligns with the traditional approach of teaching mathematics, since teaching is viewed as 'banking or implanting' and learning is seen as a passive reception of mathematical ideas and problem-solving methods. PSMT 7's use of the phrase "correct them, show them the correct methods" seem to support a behaviourist approach to teaching where learner mistakes are not tolerated.

Table 10: Conceptions of PSMTs

Question	PSMTs' responses (reason not shown)		Comments on question
	Agree	Disagree	
1	3,4,5,6,7,8,10,15,17	14,20	Focused on problem-solving strategies and problem solving
2	4,5	3,5,6,7,14,15,17	
4	3,4,5,6,7,8,10,14,15,17, 20	14	Problem-solving view
5	3,4,5,6,7,8,10,14,15,17,20		
6	3,4,5,6,7,8,10,14,15,17,20		
7	3,4,6,7,8,10,14,15,17,20	5	
8	3,4,7,8,14,17,20	5,6,10,15	
9	3,4,5,6,7,8,10,14,15,17		
10	3,4,5,6,7,8, ,14,15,17,20	10	
11	3,4,5,6,7,8,10,14,15,17,20	14	Progressive pedagogy
12	3,4,5,6,7,8,10,14,15,17,20		
13	4,6,7,8,10,14,15,17,20	5	
14	3,4,5,7,8,10,14,17,20		
15	3,4,7,8,10,15,17,20		
16	3,4,7,8,10,14,17,20	5,15	
17	7,8,10,17,20	3,4,5,15	Traditional pedagogy (instrumental and Platonist view)
18	3,4,5,7,8,10,15,17,20		
19	3,4,5,7,8,10,15,17,20		
20	4,5,7,8,15,17,20	10	
21	4,10,15,17,	5,20	
22	17	4,5,7,8,10,15	Role/responsibility of teachers
23	17	4,5,7,8,10,15	
24	17	4,5,7,8,10,15	
25	4,7,15,17	5,8,10	
26	4,17	5,7,10,15	
27	17	4,5,7,10,15	

Table 10 shows that PSMTs prefer using a variety of strategies in solving mathematical problems. An overwhelming majority of PSMTs agreed with items 4 to 10, with item 8 being an exception, since four of the PSMTs disagreed. The PSMTs unanimously concurred with the progressive problem-solving pedagogy being addressed by items 11 to 16.

For items 17 to 21, the responses of the PSMTs were in the affirmative, except for item 17, where there was a split of opinions. In contrast, the PSMTs disagreed with the traditional problem-solving conception purported by items 22 to 27. Again, the PSMTs' views were split on item 25.

5.20 Ethical Considerations

According to Cresswell and Poth (2018), ethical issues arise at each phase of any research and they argued that in qualitative research such issues crop up prior to conducting the study; at the beginning; during data collection and analysis; in reporting the data, and in publishing the study. Cresswell and Poth (2018) categorized ethical issues as respect for a person (i.e. privacy and consent); concern for welfare (i.e. minimizing harm and augment reciprocity), and justice (i.e. equitable treatment and enhance inclusivity).

In this study, ethical permission was sought and obtained from the university's ethics committee before commencing with the study. On receipt of the ethical clearance certificate, this researcher informed the university's HOD and the dean in writing of his intention to collect data from the prospective mathematics teachers and the duration of this process. Particular attention was paid to any conflict of interest as this researcher is part of the teaching staff and may have been teaching some of the participants.

In this regard, this researcher wrote and issued letters to the pre-service mathematics teachers explaining the purpose of the research, guaranteeing that activities done during the study were not linked to formal assessment and that such activities would take place outside the university's normal working hours. The participants were assured that participation was voluntary.

This researcher was aware that teaching and researching simultaneously borders on issues of power imbalances. To ameliorate the intrusive effects of a researcher, this researcher also addressed issues around freedom of participation and the anonymity of participants. In this regard, this researcher designed a consent form which all participants signed before the study commenced, stating that all activities would only be used for research purpose. Additionally, participants were informed that they had the right to withdraw at any point without giving a notification or a reason.

In this study, group responses were analysed. Thus, there was no way that the responses could be linked to individuals. The data collected for this research have been stored in devices that have passwords and the data have only been used for research purposes. The documents containing the pre-service teachers' responses did not bear any labels (names and student numbers) which could be traced to the participants.

The documents used by PSMTs were converted into an electronic format and stored in devices protected with passwords and the hard copies were shredded. Fictitious numeric names were assigned to the participants' responses (documents) where necessary. When reporting, this researcher adhered to the principle of honest reporting, while ensuring that the participants' information was not disclosed. The university has electronic and hard copies of the research report. This researcher avoided plagiarizing when compiling and disseminating the outcome of this research.

5.21 Summary

This chapter discussed the research paradigm and design of the study. The study used a qualitative research approach in exploring PSMTs' non-routine mathematical problem-solving strategies in FM. An exploratory, single case study, with embedded units, was used. This researcher also deliberated on how rigour was addressed. In section 5.11, this researcher described the participants and their contexts. Data were generated through observations, interviews, documents and an open-ended questionnaire (section 5.12 - 5.14). The data analysis processes were discussed in detail with examples from the data generated in this study (sections 5.15 - 5.19). This researcher further elaborated on how ethical concerns were addressed in this study (section 5.20).

It is worth reiterating that the data generation was supposed to have taken place in the first semester. However, due to financial constraints, data generation only began in the second semester. This meant that this researcher had to impose taking financial mathematics as a criterion for being in a group. Consequently, two main groups emerged. There were more PSMTs taking the financial mathematics module. These were further split into two subgroups four and five members, respectively. The group made up of PSMTs not taking the financial mathematics module had only two members. In the next chapter, the results are interpreted.

CHAPTER SIX: DATA ANALYSIS AND PRESENTATION OF FINDINGS

6.1 Introduction

In chapter five, this researcher elaborated on how the data for this study were generated and analysed in relation to the constructs formulated in chapter four. This chapter endeavours to relate the finding of this study to the research cited in chapters three and four. This formed the basis for finding answers to the questions posed in chapter one. The sections that follow provide an insight on how the main research question was answered.

6.2 Problem-Solving Strategies, Algebraic Activities and Strategy Variation

The main research question for this study was: How can the learning experiences of pre-service mathematics teachers be optimized in the use of strategies in solving non-routine problems in financial mathematics? The answer to the main research question has been answered by examining the following specific research questions:

1. What are the cognitive strategies that pre-service mathematics teachers use in the context of their problem solving in financial mathematics?
2. To what extent do pre-service teachers exhibit intra-task or inter-task strategy flexibility?
3. What are pre-service mathematics teachers espoused conceptions concerning non-routine mathematical problem solving?
4. To what extent do pre-service teachers draw on algebraic activities when solving non-routine mathematical problems?

In chapter 5, the data generated by observations, interviews and documents were analysed. Section 5.17 illuminated how PSMTs' written work was analysed in line with the constructs which were developed in Chapter 3 and 4. This section was concluded by providing an overall analysis of the written responses (see table 8). Similarly, PSMTs' verbatim transcripts were analysed (see section 5.18), resulting in table 9. Excerpts from interview transcripts have been included to show the written responses and observations were analysed.

To provide a comprehensive response to research questions 1 – 3, table 8 and table 9 substantiated the findings by drawing on the excerpts from the transcripts and the written responses of the PSMTs per each problem solved by them. The tables show that PSMTs used several problem-solving strategies in resolving non-routine mathematical problems in financial mathematics. The strategies used were (1) algebraic strategy, (2) numerical strategy, (3) verbal strategy and (4) pictorial strategy.

These strategies were either used alone or in combinations during problem solving. It should be noted that the PSMTs' ability to use a strategy does not imply success in solving the problem. For example, the M_ group and T_group both used numerical, algebraic and verbal strategies to solve problem 1 (see table 8) but attained different results. It seems both groups managed to make assumptions necessary to solve the problem, identified the variables and their values, and formulated a mathematical model based on the assumptions and variables (Yilmaz and Dede (2016). However, their answers to the problem did not only differ but both answers were completely different from what this researcher expected (see excerpt 10). The use of **numerical strategy** was in the form of *computation*, while the use of a *formula* and *substitution* marked the use of **algebraic strategy** (see excerpt 10). A closer look at their solution processes shows that they differed on the value of n. It seems the groups differed in their interpretation or definition of n. Polya (2014) argued that the first step in solving the problem is to understand the problem. This entails understanding the verbal statements of the problem, and identifying unknowns, the data and conditions of the problem, as well as introducing suitable notations. Introducing a suitable notation is a critical part of problem solving. It is at this stage that the problem solver should pay attention to the quantities and their units.

A closer look at M_group's work shows that the group committed a mathematical error by not writing down the unit of n and thus implying that the unit in use is years (see excerpt 10).

Excerpt 10: T_group Problem 1 Solution vs M_group Problem 1 Solution

M_Group: problem 1 solution

first option (between 1 to 10 days)
if he borrows money from the bank to get ~~3%~~ 3% off of the R2800 he owes the merchant.

$R2800 - R2800 \times 3\% = R2716{,}00$

If he borrows that money from the bank
∴ ~~S~~he will end up paying:

$SI = A(1 + in)$

$SI = R2716(1 + 1(5\%))$

$SI = 2851{,}80$

T_Group: Promblem 1 Solution

1. Discount for paying in 10 days

$R2800 \times \frac{3}{100}$ ~~= R2716,00~~

$= R84{,}00$

∴ Discount will be R84,00

∴ He/she must borrow from the bank

$R2800 - R84$

$= R2716{,}00$

He/she will borrow R2716,00 from the bank to take advantage of the discount.

T_Group: problem 1 Solution contd

If he/she borrows from the bank.

$A = P(1 + in)$

$A = R2716\left[1 + \frac{5}{100}\left(\frac{1}{12}\right)\right]$

$A = R2727{,}32$

∴ $R2727{,}32 - R2716 = R11{,}32$

At the bank there will be a charge of R11,32 after every thirty days thereof.

It was important for this researcher to understand first what this n is and how the groups arrived at its value. A member of the T_group clarified this during the interview as follows:

Excerpt 11: T_group Problem 1 Transcript

⋮	
Respondent:	Yes, we said now that we have the discount, let's see if he goes to borrow from the bank how much will he pay.
Interviewer:	How much is he borrowing from the bank?
Respondent:	He is actually borrowing this money left from the discount.
Interviewer:	The balance?
Respondent:	We said he borrows the balance and goes to pay the merchant within 10 days then no we are looking if he pays the bank.
⋮	
Interviewer:	So like, okay, okay so this man goes to the bank to borrow money, when does he borrow the money? For how long does he borrow this money according to your analysis from the group? For how long was he borrowing the money?
Respondent:	We thought he is borrowing it for 30 days.
Interviewer:	You said for 30 days?
Respondent:	Yes
Interviewer:	Okay for 30 days, in other words, I am trying to put a zero here just for argument sake. So you are saying this guy is going to borrow on the first day.
Respondent:	Yes.
⋮	
Interviewer:	So you think the bank only lends money for a full month?
Respondent:	*No, the way we thought that day maybe it is below 30 days we do not know what to put in the term there, in then if you are using a formula.*
Respondent:	*Mmmmm.*
Interviewer:	Why are you saying, maybe I should ask this, what is that n in the formula? what does n stand for in the formula that you used?
Respondent:	*The period, it stands for the period.*
Interviewer:	*The period, the period, meaning?*
Respondent:	*Yes, how long.*
⋮	
Interviewer:	It can be a day, but when you said earlier that there will be a problem at the bank (silence) so where do we stand, there will be a problem because the number of days will be less than 30 but n can be days so which one do we take?
Respondent:	Actually we did not consider the other way if he pays maybe on day 3 or 4 onwards.
⋮	
Interviewer:	It can still work, how then are you going to write your n?
Respondent:	So I have to convert the days.
Interviewer:	You have to convert days?
⋮	

For the T_group, the n in the formula is the period for which the money is borrowed (see excerpt 11, the italicized parts). Their definition is mathematically inadequate because any value of time could be substituted for n. Also, the T_group ignored the units of quantities as they substituted values in the formula which led them to commit a **mathematical error** (Da Silva et. al, 2009). The units of i and n did not match. The PSMTs' inability to define n is not surprising. The NCS – CAPS for mathematics does not have a definition for simple interest in its glossary, even though this concept is taught from Grade 7 upwards. An absence of a clear definition in the policy document is an indirect way of delegating responsibility to LTSM authors and teaching mathematicians who could use various definitions depending on their educational background.

Apart from their mathematical flaw, the T_group seemed to be unaware of the time value of money (TVM) (Gunawan & Koto, 2017) (see excerpt 11, the underlined sections). The T_group seemed to treat financial calculations without looking at the financial implications that come with time. The T_group assumed that the money would be borrowed for 30 days. They substitute a 1 in the formula implying that the money would be borrowed for the entire month. This assumption benefited the bank more than the borrower. Further, substituting one month in the formula violates the mathematical principles by having two different units for the same quantity (time) in the same solution process or calculation. The group seemed to be unaware that substituting a 1 in the formula meant that the money is borrowed for a year, since interest rate was charged per annum. The T_group was not alone in recalling a correct formula and also being unable to articulate eloquently the meaning of each variable.

The D_group used $n = 30\ days$ in their calculations while defining n to be the number of years. By substituting 30 in the formula, the D_group Just like the T_group committed a mathematical error since the units of n and i have to be the same for the formula to yield the correct result.

When asked to explain the meaning of the n in the formula, the members of the M_group gave two opposing answers and they could not justify why any of the explanations was correct (see excerpt 12, the bold text). The M_group wrote the value of n as $\frac{1}{12}$ when they performed calculations. This implies that this group defined n as the duration of the investment measured in years.

The M_group like the D_group and the T_group used of $n = 30\ days$. This did not take into account the discount period of $10\ days$. Thus all the groups committed a contextual error by using 30 days instead of 20 days.

It is important to note that the members of the T_group and M_group were taking a module in financial mathematics at the time of data generation. The inability of the groups to demonstrate a sound knowledge of financial principles (such as TVM) is worrying given that those PSMTs were in the final year of their studies.

Excerpt 12: M_group Problem 1 Interview

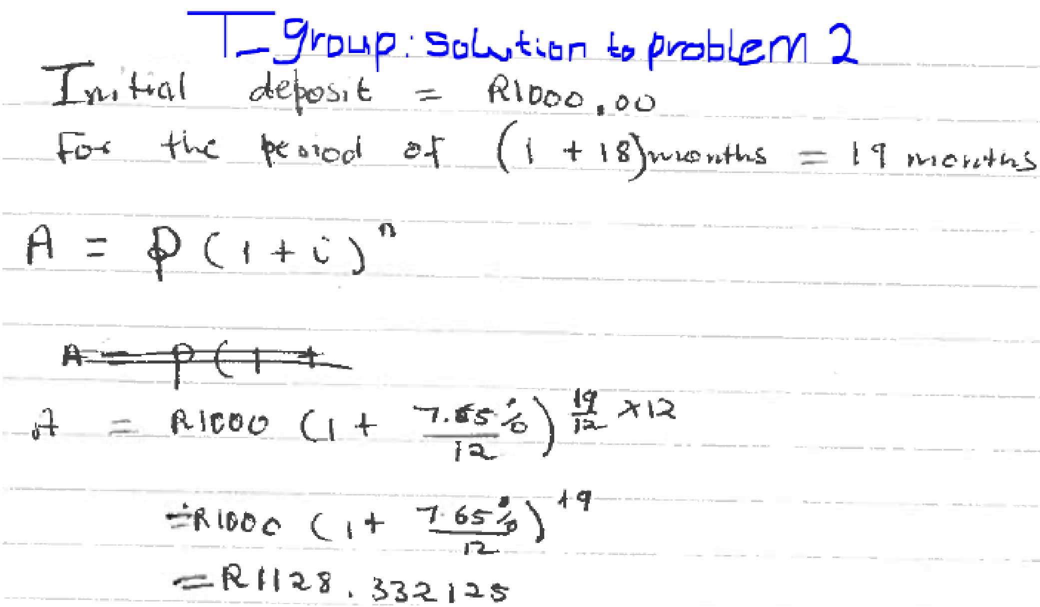
⋮	
Interviewer:	When you have a formula, you see, you are using a formula, that n what is it?
Respondent 2:	**Number of years.**
Interviewer:	Number of years?
Respondent 1:	**Months, we substituted months.**
Interviewer**:**	**Why should it be months?**
Respondent 1:	**It was specified compounded annually but it should months.**
Interviewer:	Why months? Because the question is talking about a month.
Respondent 1:	A month, yeah a month, 30 days is a month sir.
Interviewer:	**For now let's just talk about the formula without the question. When we write this formula like this, that n that is there, what is it?**
Respondent 1:	**Is number of payments.**
Interviewer:	Number of payments?
Respondent 1:	**Number of years. If it is simple interest**. If it's future value or present value it would be number of payments.
Interviewer:	So in this case, what is it?
Respondent 1:	**I think it's the number of months**, the money accumulates because it has to do with the percentage here.
Interviewer**:**	**Why should it be months? Why not hours, why not days?**
Respondent 1:	**I have not had too much knowledge about months but since I learn financial mathematics I learn that *n* is the number of years or months.**
⋮	

Although the M_group's solution process met all the requirements for using the simple interest formula and resulted in the closest answer, they also struggled with the meaning of the n and could not get the exact number of days for which the money would be borrowed. All the groups were unable to identify and use the problem conditions (Poyla, 2014).

This study has extended the findings of the Department of Basic Education (2015, 2016, 2017) on inappropriate use of formulae by showing that PSMTs are able to recall appropriate formulae but are unable to use them efficiently, due to their lack of understanding of the meaning of the variables involved and inadequate knowledge of the time value of money principle.

In responding to problem 2, the T_group and the M_group communicated their solution processes numerically and algebraically. A numerical strategy was used to find the accumulated money after 18 months, while an algebraic strategy (a formula) was used to find the future value of the annuity and the accumulated value of R1000. These two groups seemed to have made similar assumptions about the initial deposit of R1000, since both indicated that it would be invested for 19 months (see excerpt 13).

Excerpt 13: M_group Problem 2 Solution vs T_group Problem 2 Solution


T_group: Solution to problem 2

Initial deposit = R1000,00

For the period of (1 + 18) months = 19 months

$$A = P(1+i)^n$$

~~A = P(1 +~~

$$A = R1000\left(1 + \frac{7.65\%}{12}\right)^{\frac{19}{12} \times 12}$$

$$= R1000\left(1 + \frac{7.65\%}{12}\right)^{19}$$

$$= R1128.332125$$

Although both groups made similar assumptions about the deposit and chose an individual approach (IP) in solving the problem (Pournara, 2009), they differed on how they interpreted the monthly deposits of R300 (see excerpt 14). The T_group seemed to have considered the timing of the deposit, while the M_group did not. Polya (2014) posited that considering problem conditions is a critical phase of problem solving. In this problem, the context and time conditions were essential for the successful completion of the solution process. By failing to interpret the problem context and conditions, or ignoring the timing of the deposit, the M_group did not only commit a **mathematical error** (Da Silva, 2009), but also a **contextual error,** which could be very costly in the financial sector.

Excerpt 14: M_group Problem 2 Solution vs T_group Problem 2 Solution

T-group: Solution to Problem 2 Contd

For the R 300. rands deposited everymonth For 18 months.

$$Fv = \frac{x\left[(1+i)^n - 1\right]}{i}(1+i)$$

$$= \frac{300\left[\left(1+\frac{7.65\%}{12}\right)^{\frac{18}{12}\times 12} - 1\right]}{\frac{7.65\%}{12}}\left(1+\frac{7.65\%}{12}\right)$$

$$Fv = R\,5739.158841$$

M_group: Solution to Problem 2 Contd

$$Fv = x\left[\frac{(1+i)^n - 1}{i}\right]$$

For monthly deposits

$$Fv = 300\left[\frac{\left(1+\frac{7.65\%}{12}\right)^{18} - 1}{\frac{7.65\%}{12}}\right]$$

$$Fv = R\,5702.803469$$

Excerpt 14 shows the written responses of the T_group and the M_group to problem 2. The T_group have $= \frac{18}{12} \times 12$, while the M_group has $n = 18$. The difference in how the exponent has been written is of pedagogical value (benefit) as Moon, a member of the T_group explained (see excerpt 15, the bold text). Here Moon drew on his knowledge of the South African School curriculum to emphasize and legitimize a way of writing and teaching.

Excerpt 15: T_group Problem 2 Transcript

⋮	
Moon:	**It will be 19 over 12 times 12.**
Siphelele:	U 19 (It is 19).
Noluthando:	Asithi 19 (Let us say 19).
Mkhize:	The 12 and 12, they will vanish.
Siphelele:	19.
Moon:	If you say 19 they will ask you where is the 12.
Siphelele:	Ayi (Nooo).
Moon:	**If you say 19 over 12 times 12 it will be easy.**
Mkhize:	The 12 and the 12 have cancelled.
⋮	
Moon:	**If you divide here you multiply there, that that is what you tell Grade 12** (If you divide here you multiply there, that is what you tell Grade 12).
Mkhize:	Sure.
Moon:	**That's what you tell Grade 12 if you divide the interest, you have to multiply the number of n there.**
⋮	

Moon seemed to background the mathematical meaning of n, while foregrounding how learning mathematicians should be taught how to write. It is not clear whether Moon has a definition of n or not, because when Noluthando raised the issue of meaning Moon did not respond. This researcher has noted that the formula in use there has its roots in arithmetic progression, where n denotes the number of terms in a sequence. Pournara (2013a) noted that PSMTs often get confused with what n represents.

Scrutiny of the T_group transcript on problem 2 shows the solution shown in excerpts 13 - 15 was not their initial attempt at the problem. The underlined parts of excerpt 16 show that this group initially chose the account balance approach (AB) and correctly got the balance in the account after the first deposit in May. The use of the AB approach typifies how PSMTs employed **numerical strategy** in solving problem 2 (see excerpt 16). At first, they calculated the interest on R1000. This was added to R1000 to get R1006.375 at the end of the month (compounding of interest). R300 was added to the balance in the account to get R1306.375 at the beginning of May, 2008.

Excerpt 16: T_group Problem 2 Transcript

⋮	
Mkhize:	Ngizo yibeka nga macalculations, ngithi mina, ufaka R1000 ngale nyanga angithi? (I will do this using calculations, I say, you deposit R1000 at this month, right?).
Siphelele:	Faka u one plus interest yakhona u seven comma six five compounded monthly (Add one plus the interest which is 7,65% compounded monthly).
Mkhize:	For? One month? Ngisa calculator inyanga yakuqala. (For one month, I am still calculating for the first month).
Siphelele:	One month.
⋮	
Siphelele:	U 1000 wazala aseyedwa angakaqali kufaka lo 300 (The R1000 accumulated interest before he starts depositing the R300).
Noluthando:	Eeee.
Siphelele:	Sizoqale sithole iamount ezo add 300 phezulukwayo… i300 uzoqala, u 1000 asezele (We'll first find the amount to which R300 will be added, the R300 will be added after the R1000 has accumulated interest).
⋮	
Mkhize:	Separate… u 1000 u accumulate 1006.375 so after one month nayi lemali (Separate…R1000 accumulated to R1006.375 after one month here is the money).
Nathi:	Plus bese iyiphela inyanga (Plus then the month ends).
Mkhize:	Sizo deposita ama 300 (We will now deposit the 300).
Siphelele:	Kusho kuthi imali esizodeposita u1306, bani bani (meaning the money that will be deposited is 1306 comma something).
Mkhize:	It is, now. one zero zero six plus.
Nathi:	300 which is equal to 1306.375.
⋮	

Using the account balance approach is notoriously a tedious process that demands great patience from the problem solver. In using this approach, the PSMTs drew on their intuition to transit from the world of words to the world of numbers (Skemp, 1986). Pournara (2013b) posited that the AB approach is good in developing conceptual understanding and it best approximates what happens in a bank. He further supported PSMTs' use of AB approach by arguing that in "the world of banking, interest is calculated daily and compound monthly. This means that interest calculations done each day use a *daily interest rate*, and interest is calculated on the balance in the account at midnight" (Pounara, 2013b, p .91).

However, the AB approach becomes cumbersome if the exponent is large. Perhaps it is for similar observations that this group did not continue with it. It is worth noting that not all members of the group wanted to use the AB approach.

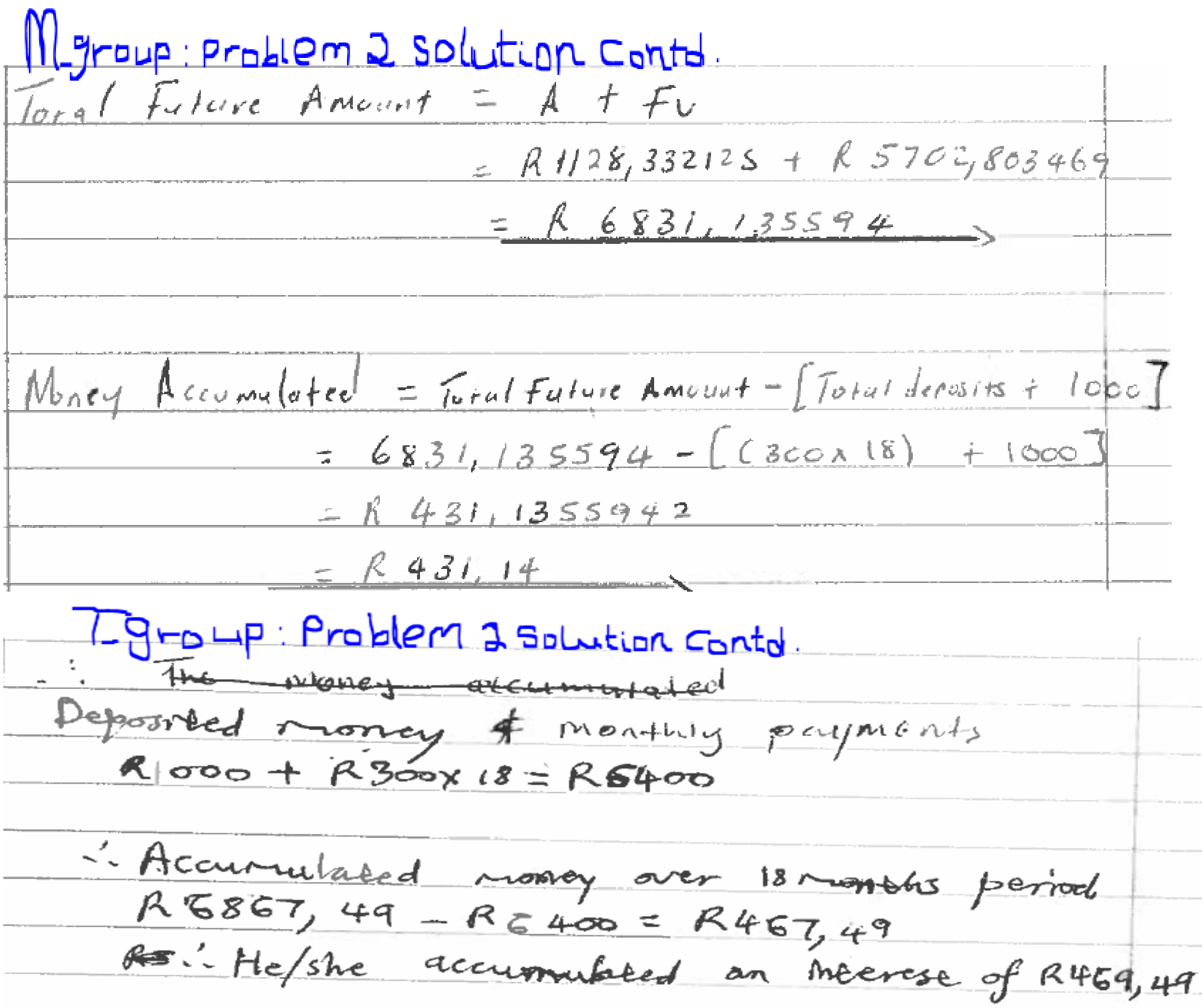
Noluthando was vehemently against the use of AB approach from the onset. She explained to the group her position as follows (attention is drawn to the underlined and italicized parts of excerpt 17):

Excerpt 17: T_group Problem 2 Transcript

⋮	
Noluthando:	*For one thousand we can, you use compound interest. Then for these monthly payments of R300, we can use future value formula for annuities.*
Mkhize:	The payments have been made for 18 months, this R300.
Noluthando:	*But the R1000 was made once so that is why I say we have to.*
Mkhize:	One thousand stayed for how many months?
Noluthando:	18 months but…
⋮	
Noluthando:	*People! Simple just use compound interest, compound interest plus ifuture value formula then you will get the correct answer.*
Nathi:	Awuzame ukuyi explaina ke, ukuthi kanjani? (Try to explain how?).
Noluthando:	*Okay, okuqala nje, sizo understander ukuthi lo uzo ngena kuqala angithi?, lo angaka ngeni (Okay, firstly we will understand that this one will be deposited first, before this one)*.
Nathi:	Mmmmm.
Noluthando:	*Uma engakangeni kodwa, siyazi kuthi uzo hlala iperiod kabani? Ka 18 months but angeke e fakwe inyanga ne nyanga (If it not deposited but we know it will stay for how long. For 18 months but it won't be deposited every month)*.
Nathi:	Ngicela ukubuza futhi kancane, kukhona into engifuna iqake la, u300 bazo depositwa for 18 months angithi? (Can I please ask, there is something which needs to be clear here. The 300 will be deposited for 18 months right?)
Noluthando:	*Yes, bona abe future value, bazo ngena kwi formula ya future value eceleni (Yes, they are for the future value, they will be in the future value formula)*.
⋮	

The solution to problem 2, in both groups (T_group and M_group), seems to contain **linguistic errors.** For example, the phrase “accumulated money” seems to have the same meaning as “interest”, based on the groups’ solutions (excerpt 18).

Excerpt 18: M_group Problem 2 Solution vs T_group Problem 2 Solution


M_group: Problem 2 solution contd.

Total Future Amount = A + Fv

= R 1128,332125 + R 5702,803469

= R 6831,135594

Money Accumulated = Total Future Amount − [Total deposits + 1000]

= 6831,135594 − [(300 x 18) + 1000]

= R 431,1355942

= R 431,14

T_group: Problem 2 Solution contd.

∴ ~~The money accumulated~~

Deposited money & monthly payments

R1000 + R300 x 18 = R6400

∴ Accumulated money over 18 months period

R6867,49 − R6400 = R467,49

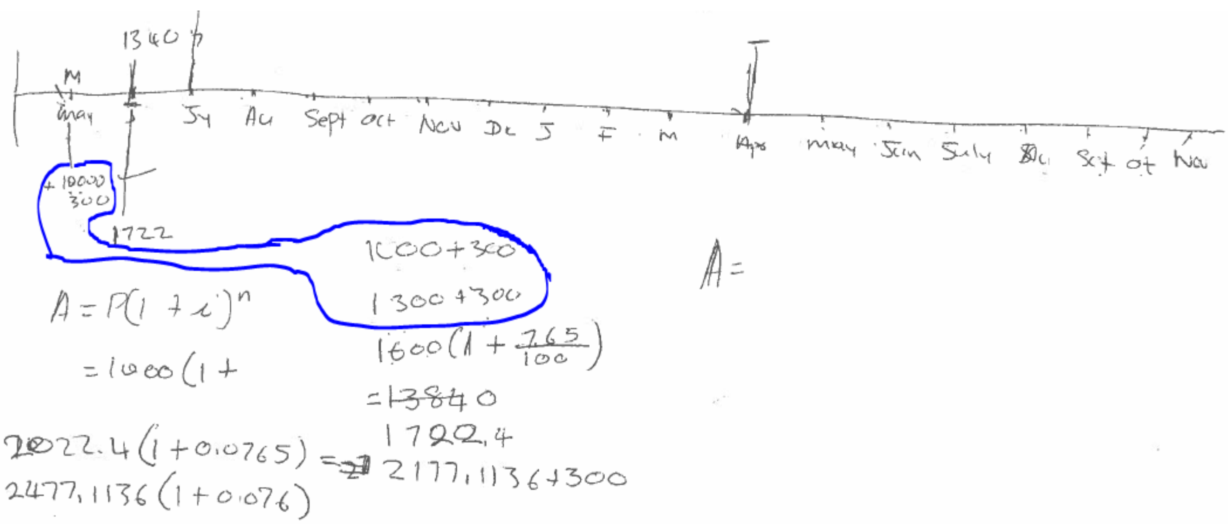
∴ He/she accumulated an interest of R469,49

Their interpretation of “accumulated amount” is contrary to the definition provided by Jenkin et al. (2014) for Grade 10 Learning mathematicians. The Department of Basic Education has, over the years, noted with great concern the lack of mathematical vocabulary in the language of financial mathematics displayed by Grade 12 learning mathematicians during examinations. The DBE has urged teaching mathematicians to pay attention to linguist issues when teaching financial mathematics so as to enhance learning mathematicians’ performance in high stake examinations. The importance of fluency in the language of learning and teaching (LOLT) was underscored by Setati

(2005), who pointed out that language is key in developing a conceptual understanding of mathematical concepts and skills and initiating learning mathematicians into ways of communicating mathematically.

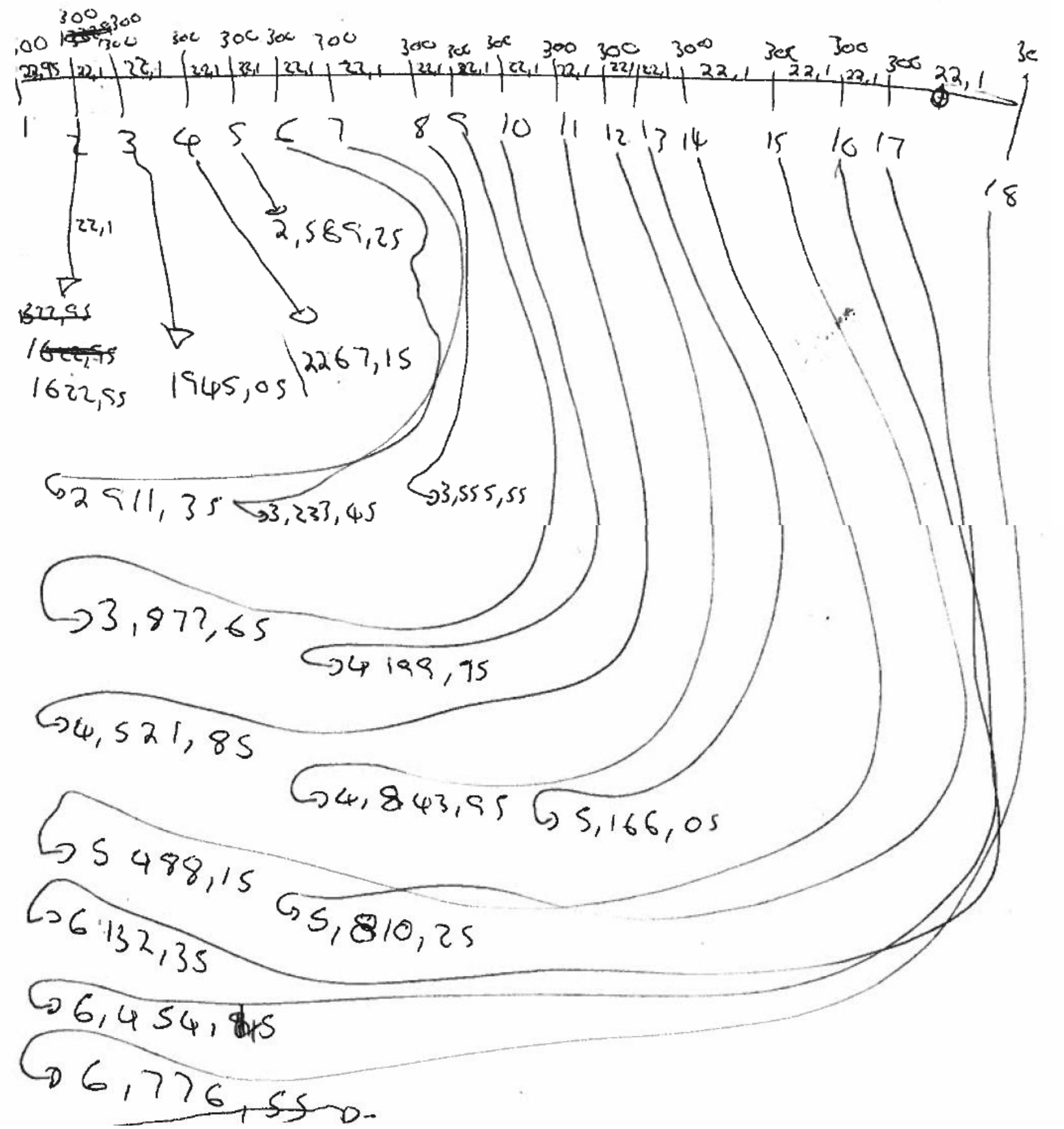
Unlike the T_group and M_group, which used an IP approach in solving the problem 2, the D_group first used an account balance approach with three different strategies. The first strategy was a pictorial strategy in a form of a timeline, accompanied by a formula (see excerpt 19).

Excerpt 19: D_group Problem 2 Solution



As argued by (Polya, 2014), problem solvers should endeavour to understand the problem. He suggested that problem solvers should draw a diagram and label it appropriately, using suitable notation. Aligning themselves with Polya's heurist, the D_group used the names of the months on their timeline. It is not clear whether the divisions on the line marks the beginning or the end of the month. However, one can infer that this group is working with an assumption that R1000 is deposited at the end of April and by convention it does not earn interest (Pournara, 2013c) (see excerpt 19, the circled part). The circled part of D_group's work further shows that this group used the AB approach, supported by a numerical strategy. In transiting from May to June, they missed the concept of the time value of money (TVM), since they only added R1300, without its interest, to R300, thus committing a *contextual error*. They did two more calculations (for July and August) but this time they considered interest earned (i.e. they calculated and compounded interest correctly). The D_group seemed to have realized the limitations in their work and decided to restart the solution process as follows:

Excerpt 20: D_group Problem 2 Solution



Excerpt 20 shows that the group continued to use the AB approach, with a numerical strategy, in resolving the problem. The timeline now shows the 18 deposits of R300. The PSMTs then calculated interest for each deposit (see excerpt 20). They then applied addition to the numbers on the timeline to get the answer. It seems as if the group assumed that the R1000 was deposited at the end of April, since there is no evidence of its interest having been computed at any point in the group's solution. The group did not compound interest in their calculation. This group disregarded two fundamental principles of finance, namely the time value of money and the compounding of interest over time. While their approach resembles what happens at the banks daily (Pournara, 2013), it also shows their lack of knowledge and understanding of banking practices and principles.

In its calculation of interest on R300, the D_group did not convert the annual interest rate of 7.65% p.a. compounded monthly, into an interest rate per period (i.e. monthly interest rate), and thus committing a ***mathematical error***. A closer look at the timeline shows that the first R300 earned an interest of R22.95, while the subsequent R300 earned R22.1 in interest over the same period. The D_group's use of the timeline is problematic as it does not clearly show whether the payment was made at the beginning of a month or at the end.

Pournara (2013c) submitted that this problem could be avoided by using a timeline in conjunction with the t-notation, where T_0 denotes the beginning of the investment interval. Using the timeline in this way could have enabled the D_group to consider the initial R1000 deposit and the subsequent R300 monthly deposit. While this group was able to identify the number of cash flows and financial variables involved, they failed to identify the appropriate time value of money technique necessary to resolve the problem (Jalbert et al., 2004; Xudong & Mingxing, 2014).

In responding to problem 3, the T_group and the M_group used algebraic strategies in the form of formula for calculating the future value for investing money for two years and the formula for finding the present value of an ordinary annuity, as well as the numerical strategy indicated by computation of the balance (see tables 6 and 7). These groups committed mathematical error(s) in the process by substituting the incorrect values into the formulae. For example, the T_group seemed to have misinterpreted or overgeneralized the phrase "5% p.a. compounded semiannually" since they made semi-annual repayments of the debt, instead of investing the loan amount for four semi-annuals and making a single payment in two years' time. Solving the problem this way led them into committing a linguistic error. Excerpt 21 (the underlined text) illuminates the linguistic struggles that Moon and Mkhize went through as they searched for the solution to problem 3.

Excerpt 21: T_group Problem 2 Transcript

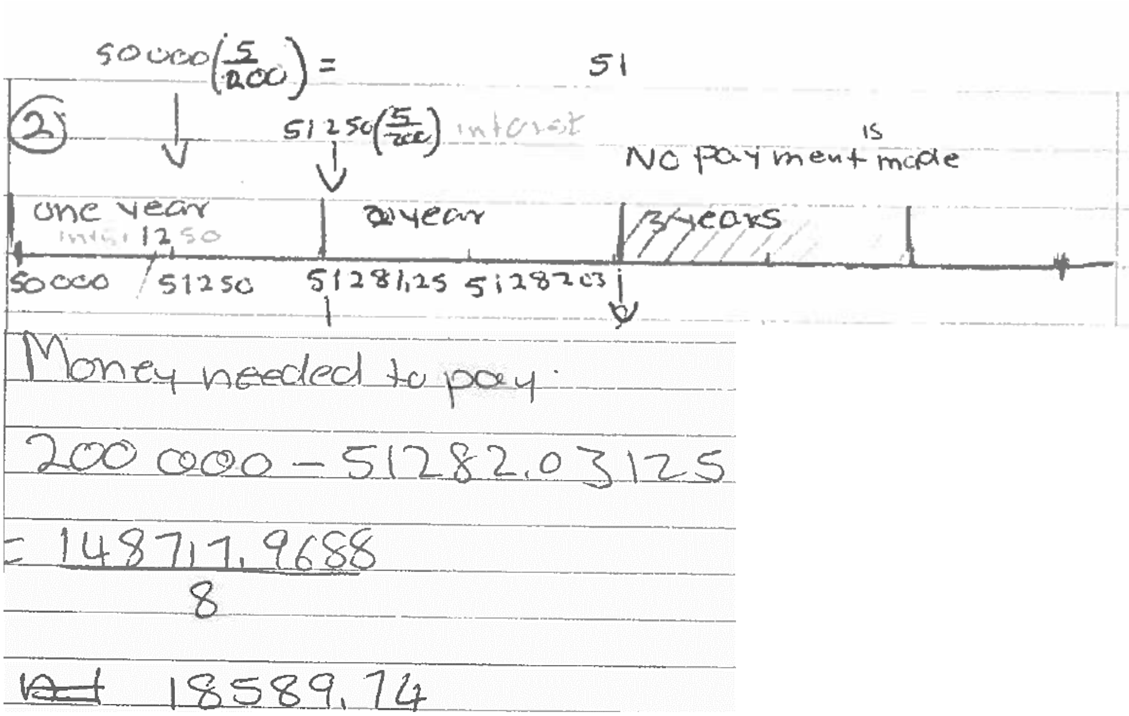
⋮

Moon:	Lena mfethu ipresent value ye annuity angazi noma awuyibone. Awuyibone lento le? Nawo amapayment ka R50 000 semi-annual, ngonyaka ukhokha kabili, uyakhokha kodwa ngonyaka, okusho ukuthi ipresent value… ngoba vele... shothi, mina ngithatha nje. Angithi imali yesikweletu le, kusho kuthi uyibolekile, angazi phela (This one brother, can you see? It's present value of annuity. Here is the payment of R50 000 semi-annual, so he pays twice a year. So it is the present value … I took it as his debt).
Mkhize:	Lo muntu usekhokhile le mali wayiqeda in two years (this guy paid his debt and finish in two years).
Moon:	Eeeeehh? (Huh?)
Mkhize:	Le ayikweledayo. Thina sifanele sazi ukuthi ama interest wakhe ebe ngakanani (B interjects) (His debt. We have to know how much was his interest (B interjects).

⋮

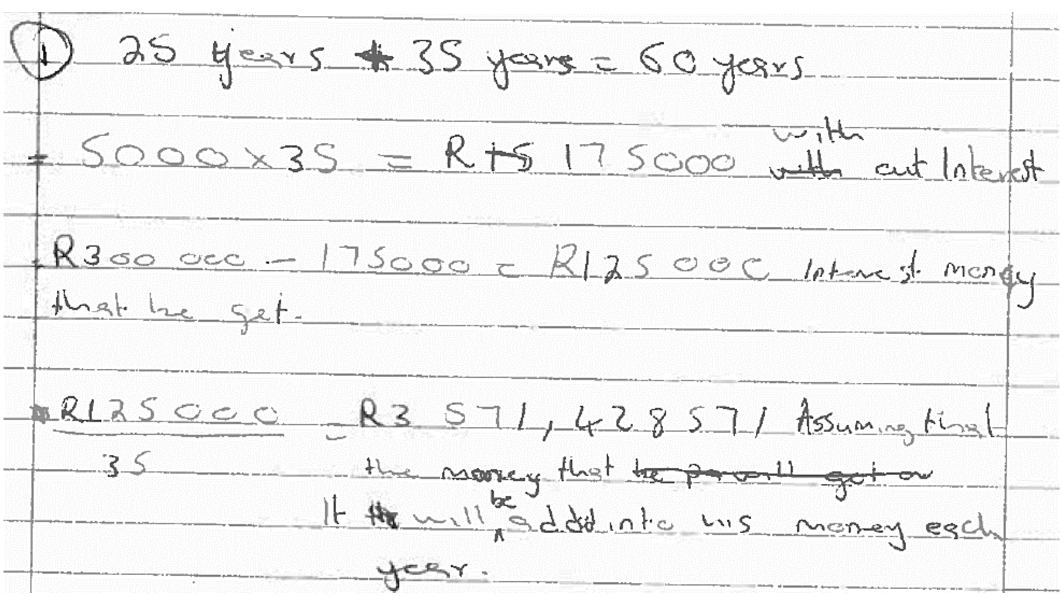
In contrast, the D_group used numerical and verbal strategies in solving problem 3. A numerical strategy was used in conjunction with a timeline (see excerpt 22), while a verbal strategy was used in defining mathematical terms such as semi-annual. It is important to note that while this group was able to draw a diagram, they were unable to show on the timeline all the information necessary for resolving the problem. For example, they did not show the loan amount of R200 000. As a result of this, they failed to relate the unknowns to the known. The absence of R200 000 led them into applying the TVM principle on the incorrect amount (see excerpt 22). The R50 000 was incorrectly labelled on the timeline. It was placed at the beginning of year 1, which implied that it was being invested. This group seemed to use their knowledge of mathematics out of context when finding a regular semi-annual payment. They used equal sharing of the balance instead of finding the present value of the eight equal payments (discounting the balance). The group committed a **contextual error** by failing to consider the TVM principle. This group's use of timelines in solving financial problems demonstrated a proficiency in **representational activities of algebra** (Kilpatrick et al., 2001).

Excerpt 22: D_group Problem 3 Solution



The D_group solved problem 4 out of context, that is, the financial aspects of the context were not used while the mathematical aspects were fluently executed (see excerpt 23). The group eloquently justified their work. The strategies used were numerical strategy in the form of computations and verbal strategy in a form of logical reasoning (see excerpt 23).

Excerpt 23: D_group Problem 4 Solution



Excerpt 23 supports this researcher's view that the D_group lacked a knowledge of compounding of interest, a knowledge of multiple cash flows and the TVM principle (Jalbert et al., 2004; Xudong & Mingxing, 2014). This shows that a knowledge of mathematics alone is not sufficient to solve contextualize problems, particularly finance problems, which are dependent on the TVM principle.

Polya (2014) advised that problem solvers should construct not only a mental representation of the problem but also a diagram to deepen their understanding of the problem. This researcher noted, on viewing the video recordings, that PSMTs resorted to working individually whenever they had a difficulty in solving the problem and only discussed their findings with the group when they thought they had found a way of solving the problem, i.e. a discovery. It should be noted that members of the D_group were not taking a financial mathematics module during this period.

While the members of the D_group were able to define and use mathematical terms, such as semi-annual, they were oblivious of the contextual principles, such as TVM and compounding of interest, necessary for the application of mathematical operations in finance (see the underlined text in excerpt 24). This shows their lack of proficiency in **contextual activities of algebra**, while being aware of **linguistic activities of algebra**.

Excerpt 24: D_group Problem 4 Transcript

⋮

Interviewer: No, no, but they are saying for the next 35 years. He was 25 years then they are saying for the next 35 years.

Respondent: Yeah, yeah, until he gets to 60 and then we think this guy, let me go back here, the first thing I have done, I said 5000 times, 35 because this guy is paying 5000 each year, at the end of the year he pays the money, this money. 35 times 5000 then I get the amount. What is left in this amount is the percentage. Then (laughs) what I have done is that I forgot where is this…

Interview: The 300 000 is here. He will get this money when he is 60.

Respondent: Yes, then I go back, I minus. I said 300 000 minus this money.

Interviewer: The money he deposited.

Respondent: Then I got this money R125 0000. Then I said now I have the amount. This is the amount that will give me the percentage because it's the amount that has the percentage. Then I go back and say this money, divided by 35 years, because this is years (laughs), then I got this money for a year. Then I was like eish!! Now I have a problem, how do I deal with this one. I asked Mosotho, now I have this money, what must I do with this money? Because it is a percentage, this money will give me a percentage of a year. This guy said, we have to do what we have to do, which is start with 5000, then we know if… eeeerrrr… let me write like this, a percentage of 5000, we say 5000 times I divided by 100, then we cross multiply. Then we have this money, then when I cross multiply. I get this one.

Interviewer: Does it make sense this percentage?

⋮

The member of D_group (see underlined text in excerpt 24) can articulate the process that led to the answer and used correct mathematical operations and principles. However, he failed to consider the financial aspects and context of the problem particularly the compounding of interest. This indicates that knowledge of mathematics alone is not adequate to solve non-routine mathematical problems in financial mathematics.

In contrast, the T_group employed algebraic strategy in the form of 'recall a formula, substitute and solve for the unknown'. This approach proved to be more problematic than the problem as it confronted the group with the equation

$$60i = (1+i)^{35} - 1.$$

At this point, the group incorrectly applied the binomial theorem to expand the bracket but made no progress and abandoned the method.

Seemingly, the members of the T-group were aware that the deposits would make a geometric series, whose sum is R300 000, but failed to work out how to find the annual interest rate in use.

Similarly, the M_group used an algebraic strategy and came across the same problems as the T_group. Although the T_group and the M_group started the solution process correctly, they failed to solve problem 4 like the D_group.

These groups were able to engage in transformational and contextual activities of algebra to some extent. These groups recalled or formulated an equation, substituted values for the known quantities and manipulated the resulting equation, to find the unknown quantity. This illustrates their proficiency in transformational algebraic activities (Kilpatrick et al., 2001). They formulated expressions such as $5000 + 5000(1 + i)^1 + 5000\ (1 + i)^2$ to show multiple cash flows, identified the problem context and applied relevant principles, such as TVM and compound interest. In this way, the T_group and M_group members demonstrated an awareness of the context in which mathematics is being applied.

Algebraic and numerical strategies were the most commonly used strategies, while verbal and pictorial strategies were sparingly used by the groups. None of the groups used a graphical strategy. The absence of a graphical strategy could be an indication of PSMTs' unfamiliarity with the use of graphs in determining solutions to algebraic equations or financial problems. At school level, learning mathematicians are given simultaneous equations and told to find the values of the variables by sketching a graph. Problem 4 seemed to require PSMTs to start from a single equation to a system of equations for its resolution or applying graphical strategies which seemed missing in the groups.

As they resolved the problems, the PSMTs grappled with representational algebraic activities (e.g. timelines), transformational algebraic activities (e.g. substituting values), contextual (timing of a deposit or payment) and linguistic (defining terms like semi – annual) algebraic activities. They also used algebraic, numeric, pictorial and verbal problem-solving strategies.

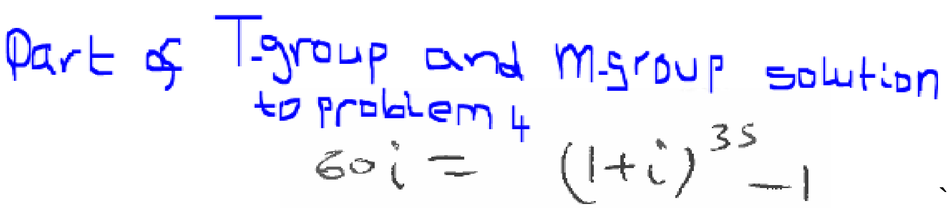
The rows in table 8 and table 9 show the problem-solving strategies and algebraic activities used by each group in solving the mathematical problem, while the columns of these tables show problem-solving strategies and algebraic activities used across problems and groups. Hence, it is concluded that the PSMTs have an extensive knowledge of problem-solving strategies and can vary these strategies.

Like the experts in Star and Newton (2009) study, the PSMTs were able to vary their strategies depending on the situation. However, the changes made by the PSMTs did not emanate from realizing a more efficient strategy, but from an impasse in the solution process.

In solving problem 4, the T_group and M_group persisted in using algebraic strategy although they seem to be working without any progress. For example, the M_group used a formula but when they could not get to a satisfactory solution, they formulated a geometric series, which eventually simplified to the same formula of ordinary annuity. These groups, while they seemed comfortable in determining the regular payment in problem 3 by performing algebraic transformations, they seemed not to be able to work in reverse. In solving problem 4, they formulated a pair of equations which could not be solved by an elimination/substitution method (see excerpt 25).

The groups' perseverance in using algebraic manipulations instead of rewriting these as two separate equations of the form $y = ax + c$ and $= ax^n$, which could be solved through graphical means, indicates the PSMTs' strategy inflexibility or lack of experience in using graphical problem-solving strategies.

Excerpt 25: T_group and M_group Part of Problem 4 Solution


Part of T-group and M-group solution to problem 4

$60i = (1+i)^{35} - 1$

Kolovou et al. (2011) and Torbeyns et al. (2009) argued that instruction on the use of a problem-solving strategy and/or experiencing success in using a particular strategy are the sources of inflexibility in problem solving. Although the PSMTs were prompted to use different strategies when solving problems, they could not seem to reconcile their answers when they arrived at diverging answers (see M_group solution to problem 1 in Appendix H).

The PSMTs seemed to be mathematically proficient since they committed fewer mathematical errors (Da Silva et al., 2009) but they appeared to lack the means to apply this knowledge and committed a number of contextual and linguistic errors in solving non-routine problems in FM. This study has supported and extended the findings of Da Silva et al. (2009), by showing that even PSMTs have difficulties in solving non-routine mathematical problems in FM, due to their inadequate knowledge of financial principles and their inappropriate use of mathematical language.

6.3 Data Interpretation: PSMTs' Conception about Problem Solving

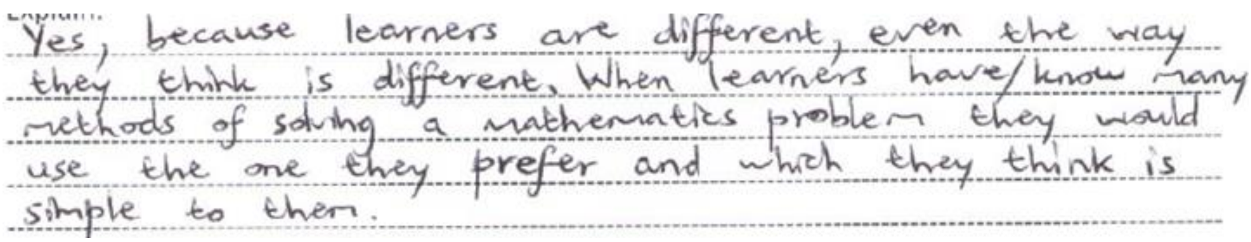
This section examines the fourth research question which guided this study: What are pre-service mathematics teachers' espoused conceptions concerning non-routine mathematical problem solving? To adequately respond to this research question, this researcher has drawn and elaborated upon the analysis provided in table 10.

An open-ended questionnaire was administered to PSMTs who volunteered to partake in this study. There were 27 items on the questionnaire (see Appendix I). The questionnaire items sought to establish the PSMTs' conception(s) of mathematical problem solving. As part of the analysis, the items on the questionnaire were placed into four clusters: 1) traditional conceptions of problem solving (which is characterized by an instrumentalist or Platonist view about the nature of mathematics); 2) traditional conceptions of problem-solving pedagogy/roles of teaching mathematicians in a traditional problem solving instruction; 3) progressive conceptions of problem solving (which is underpinned by problem-solving views), and 4) problem solving pedagogy/roles of teaching mathematicians in progressive problem-solving instruction.

6.3.1 Progressive conception of problem solving

Based on their responses to question 1 and 2, this cohort of PSMTs seemed to perceive problem solving as knowing and being able to use, effectively and efficiently, a variety of problem-solving strategies in solving mathematical problems. They argued that this would give learning mathematicians the latitude to flexibly select or combine strategies in resolving a mathematical problem. This response concurred with what this researcher noted on their written responses to the four non-routine mathematical problems posed to them.

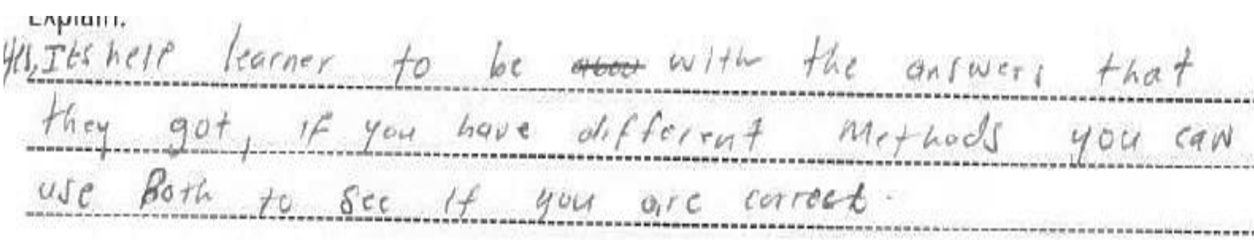
In attempting to resolve these problems, the PSMTs flexibly called on various problem-solving strategies. In support of using a variety of problem-solving strategies during mathematics learning, when asked, “Is it important to know many methods (ways) to solve a mathematics problem?”, PSMT 6 responded:


Yes, because learners are different, even the way they think is different, When learners have/know many methods of solving a mathematics problem they would use the one they prefer and which they think is simple to them.

This response indicates that PSMT 6 maybe having a socio-constructivist conception of problem solving, as he or she seemed to be thinking of individual differences in a learning environment. PSMT 6’s view is consistent with Beswick (2005a, p. 44), who asserted that “(a)not all students will progress toward instructional goals as expected; (b) there are different pathways to mathematical understanding; (c) different individuals have different mathematical understandings”.

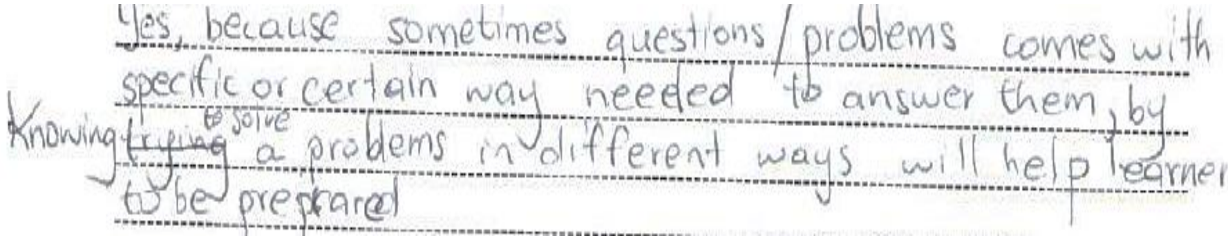
PSMT 17 viewed the use of multiple problem-solving strategies as a means of validating their mathematical processes, and thus, enhancing one’s confidence.

PSMT 17 responded:

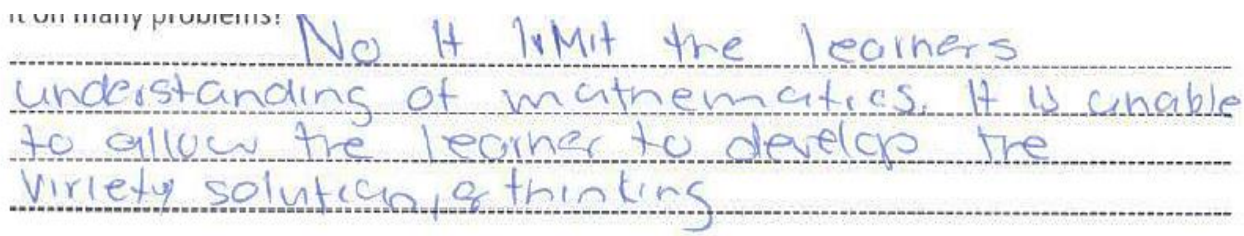

Yes, It's help learner to be ~~about~~ with the answers that they got, if you have different Methods you can use Both to see if you are correct.

PSMT 17’s view seems to align with ‘looking back’, like in Poyla’s (2014) four-step problem-solving heuristic. It is at this stage that problem solvers ask whether the result could have been attained using alternative ways, and consider the consistency of their solution, as they vary their problem-solving strategies.

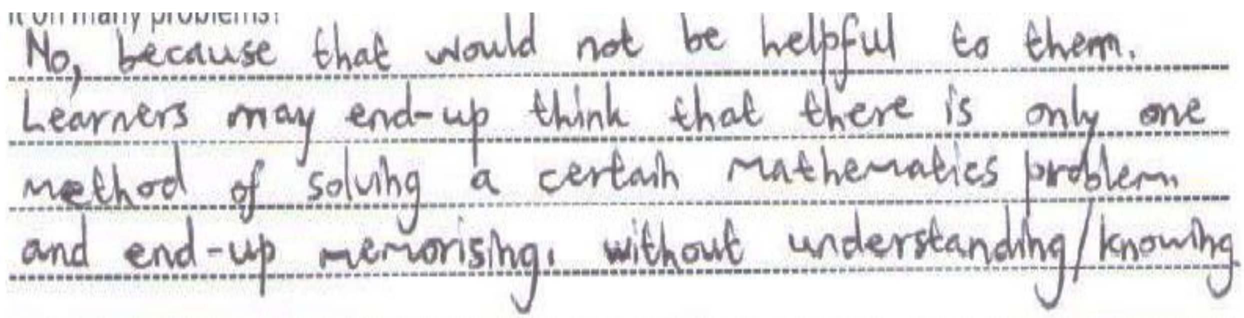
Expanding upon this point, PSMT 8 drew on his or her schooling experience to argue that using different problem-solving strategies prepares learning mathematicians for tests and examination situations, which might have some restrictions on how to respond to problems.


Yes, because sometimes questions/problems comes with specific or certain way needed to answer them, by Knowing ~~trying~~ to solve a problems in different ways will help learner to be prepared

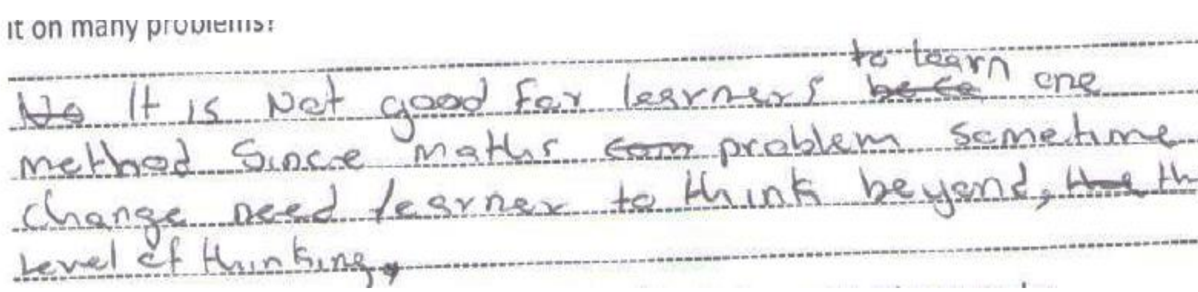
When the PSMTs were asked "Is it good for learners to learn one method of solving a mathematics problem but practice it on many problems? Explain." most PSMTs disagreed indicating:


it on many problems? No It liMit the learners understanding of mathematics. It is unable to allow the learner to develop the virlety solution, & thinking

(PSMT 3)


it on many problems? No, because that would not be helpful to them. Learners may end-up think that there is only one method of solving a certain mathematics problem and end-up memorising without understanding/knowing

(PSMT 6)


it on many problems? No It is Not good for learners ~~be~~ to learn one method Since maths ~~com~~ problem sometime change need learner to think beyond, ~~the~~ their level of thinking.

(PSMT 7)

> NO, If the method that a learner is taught to use when solving problems is complicated, it make it difficult for learners to have a clear understanding on how to answer questions.

(PSMT 17)

The responses of PSMTs 3, 6, 7 and 17 demonized teaching one problem-solving strategy and described that as an insufficient way of educating novice problem solvers. They argued that it was more likely to inhibit the development of mathematical problem-solving proficiency and manner of thinking (PSMT 3). Learning mathematicians may develop negative habits of learning, such as thinking there is only one way of solving problems, memorization, and even developing inert mathematical knowledge (PSMT 6). PSMT 6's view resonates with the findings of Boaler (2002), that learning mathematicians who were taught using traditional methods could not adapt their problem-solving strategies when confronted with non-routine mathematical problems, whereas those taught using a problem-solving approach could adapt their strategies with ease.

Alternatively, four PSMTs agreed with item 2 on the questionnaire. They seem to be more focused on success in school mathematics, as indicated by PSMT 14's response:

> Yes because the more they keep on practising the same method they will get use to it; improve the way they do it. That can help them to apply that method correltly in a test or exam.

However, PSMT 5 seemed to be aware that repeated use of one problem-solving strategy has limited benefits, even within school mathematics:

> Yes but sometimes can affect the learner if it is not not working in a problem he or she tries to answer

PSMT 5's observation shares the same sentiments with Schoenfeld (1988), cited in Boaler (1998), who contended that problem-solving strategies, aimed at resolving problems posed in school textbooks, tend to encourage development of superficial procedural knowledge, that is limited in resolving non-school mathematical problems. This cohort of PSMTs concurred that learning mathematics should significantly advance learning mathematician's constructions of problem-solving strategies beyond cognitive restructuring, i.e. adding to or adjusting current constructions.

6.3.2 Progressive conception of problem-solving pedagogy

Items 4 – 10 on the questionnaire sought to determine the PSMTs' conception of a problem-solving instruction in mathematics. Table 10 shows that PSMTs subscribed to a progressive conception of problem solving in teaching, since the majority agreed with the items. However, four PSMTs (5, 6, 10 and 15) disagreed with item 8 which read, "Knowing how to solve a mathematics problem is as important as getting the correct solution". It is important to note that these PSMTs seemed to disagree with equating "getting the correct answer" with "knowing how to solve a mathematics problem".

PSMT 6 and PSMT 10, respectively, shed more light on this disagreement by saying:

PSMT 6 responded:

Knowing how to solve a mathematics problem is as important as getting the correct solution.

I disagree, getting the correct solution does not mean you understand how to solve a mathematics problem. You can get the correct solution with wrong steps by copying someone's answer but knowing how to solve a mathematics problem gives you the capability to solve any other similar mathematics problem by applying what you know.

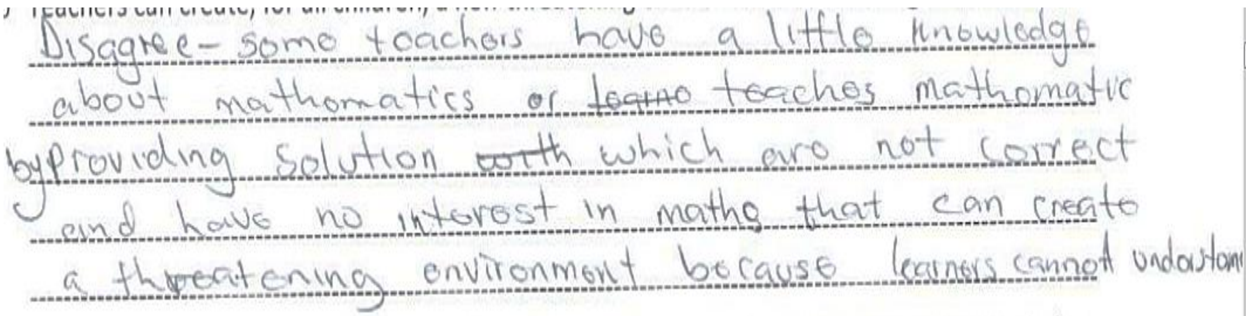
PSMT 10 responded:

> I disagree.
> The importance is on knowing how it is done, and getting a correct answer is actually the result. But even if the answer is from due to some mistake, it does not necessarily mean that you don't know how it is done.

Implicit in PSMT 6's response is the belief that the process (understanding the underlying principles, concepts and their relationships) is more important than the product. PSMT 10 is very explicit on this. This is in direct contrast to traditional teaching, which focuses on teaching standardized procedures (e.g. performing long division) and the use of formulae, such as $F_v = x\frac{(1+i)^n-1}{i}$, which in the main are useful in solving particular problems.

While algorithms are important in mathematics, learning them superficially leaves the learning mathematician unprepared to solve non-routine mathematical problems and to apply their knowledge beyond the walls of the classroom. PSMT 6's and PSMT 10's observations are not unique. Schoenfeld (1988) made similar observations about how mathematics is learned. He argued that a mastery of discipline procedures does not imply a mastery of underlying concepts or principles. In his contention, he argued that a learning mathematician who responds to problems, such as $\frac{123+123+123+123+123}{5}$ or $\frac{791+791+791+791}{4}$, by performing the laborious basic operations and gets the correct answer, would have demonstrated a procedural understanding of mathematics. This contrasts with a learning mathematician who recognizes that repeated addition is equivalent to multiplication, and division is the reverse of multiplication and, eventually, use these facts to get the answer. Similarly, in this study, this researcher has noted, that in responding to problem 1, all the groups failed to get the expected answer because the PSMTs seemed to lack an understanding of an underlying principle, i.e. the time value of money (TVM) principle, even though two groups recalled the correct formula $F_v = P(1 + in)$. In addition to their inadequate knowledge of the TVM principle, the members of the two groups were unable to appropriately define and identify the values of some of the quantities involved in the formula $F_v = P(1 + in)$.

Items 11 – 16 on the questionnaire addressed progressive pedagogy in mathematics. Almost all PSMTs' responses were in the affirmative for these items, except for PSMT 5 and PSMT 15, who disagreed with item 16. PSMT 15 did not give a reason for the disagreement but PSMT 5 seemed to draw more on experience to argue that some pedagogies are cognitively stressful to learning mathematicians.


Disagree – some teachers have a little knowledge about mathematics or ~~teamo~~ teaches mathematic by providing solution ~~with~~ which are not correct and have no interest in maths that can create a threatening environment because learners cannot understand

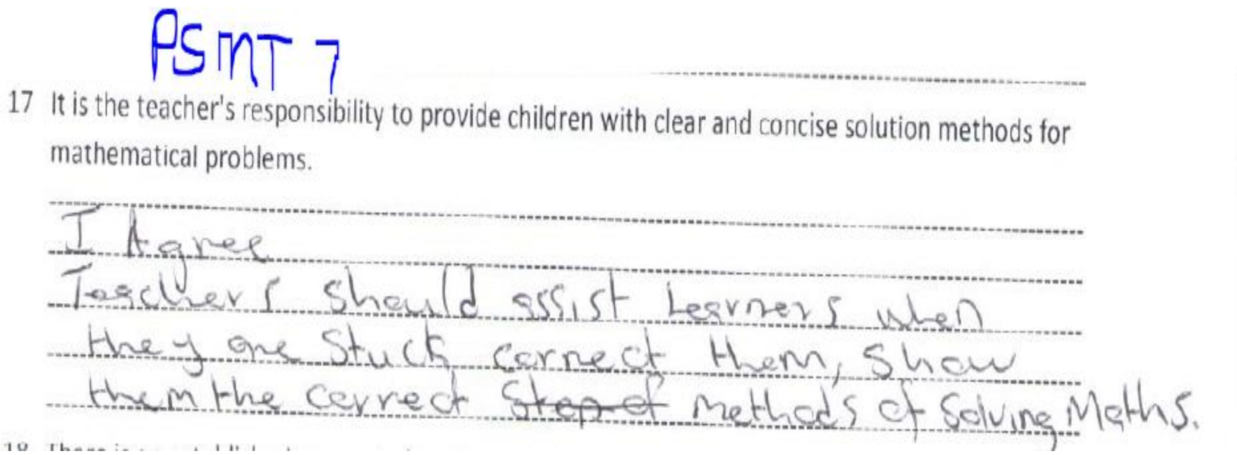
The concerns of PSMT 5 have also been noted by the Department of Education (2011c, p. 62) which argues that:

> *Many local and international studies have shown the existence of a set of attitudes, described as 'mathsphobia', in school-going learners and in the population at large… it is the responsibility of the teacher to endeavour to win learners to mathematics…. not formalising into the abstract prematurely but first taking care to develop understanding and process skills... The interests of all needs to be taken into account in providing access to mathematics.*

6.3.3 Traditional conception of problem solving

Items 17 – 21 sought to establish the extent to which PSMTs subscribe to a traditional conception of problem solving. The PSMTs were in support of all items except item 17. This item seemed to place the teaching mathematicians at the centre of learning and make the teachers a sole arbiter of learning. Item 17 assumes that it is the teacher only who can provide clear and concise solution methods.

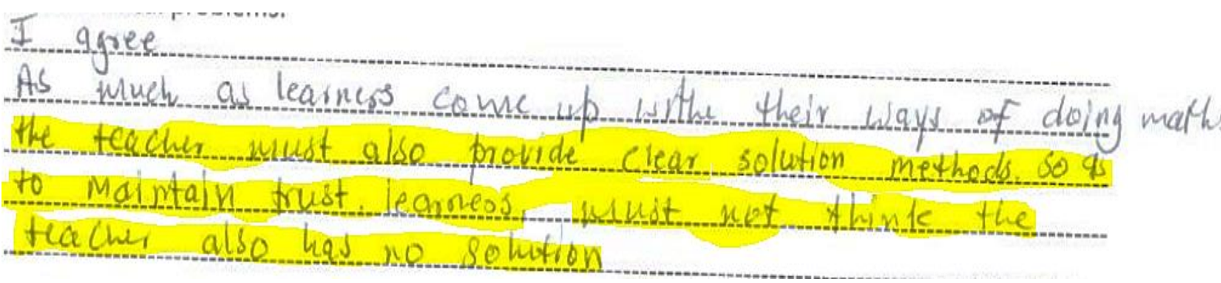
PSMTs' views were split on item 17 while they agreed with item 18 - 20. This researcher can only infer that the difference of views on item 17 could be attributed to how PSMTs' perceive their roles in the teaching and learning of mathematics rather than them seeing learning mathematicians as capable or incapable problem solvers. PSMTs who believed that it is their responsibility to provide correct methods or answers to learning mathematicians supported item 17. For example, PSMTS 7 said:


PSMT 7

17 It is the teacher's responsibility to provide children with clear and concise solution methods for mathematical problems.

I Agree
Teachers should sssist Learners when
they are Stuck correct them, show
them the correct ~~Step of~~ methods of Solving Maths.

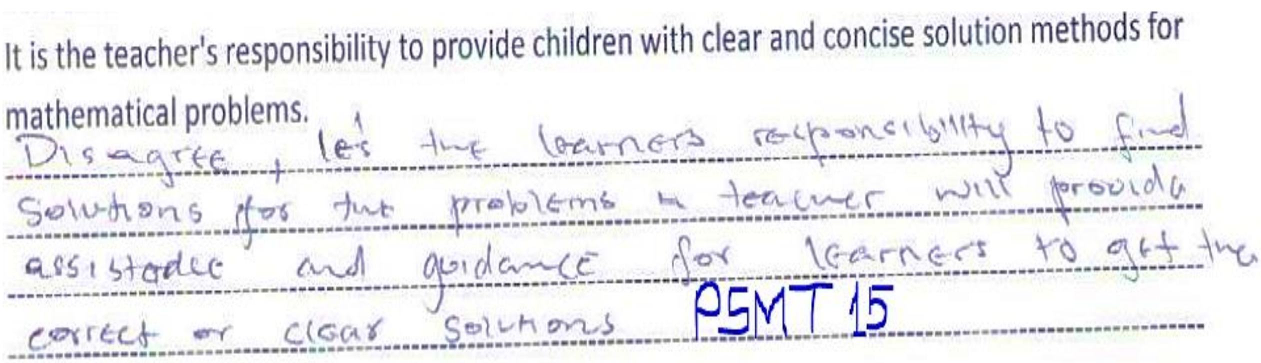
In responding to item 17, PSMT 10 admits (see response) that learning mathematicians can construct efficient problem-solving strategies. However, PSMT 10 regards learning as the responsibility of the teacher who should demonstrate knowledge at every possible opportunity. PSMT 10 responded:


I agree
As much as learners come up with their ways of doing maths
the teacher must also provide clear solution methods so as
to maintain trust, learners must not think the
teacher also has no solution

PSMT 10 viewpoint may demonstrate conflicting conceptions of mathematical problem solving. PSMT 10 seems to subscribe to a progressive problem-solving approach which informs the centrality of the learning mathematician in the learning process and a traditional problem-solving approach. Proponents of a progressive conception of problem-solving regard mathematics as a human activity (Fanzan 2002; Gravemejeir, 1999; Department of Basic Education, 2011c). An activity in which novice problem solvers should gradually and progressively be engaged to enhance their mathematical reinvention. Beswick (2005) argued that, from a constructivist perspective, the role of a teaching mathematician is to facilitate learning mathematician's constructions of problem-solving strategies, which should significantly go beyond adding to and adjusting existing constructions.

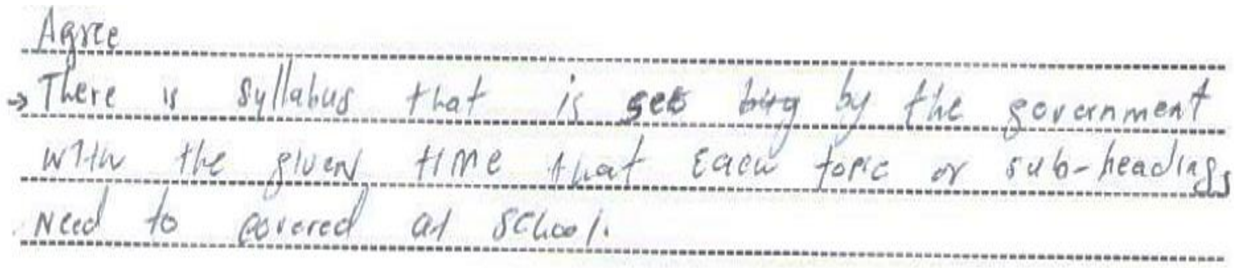
PSMT 10 appears to doubt the quality and possibly usefulness of problem-solving strategies generated by learning mathematicians as she or he argued that the teaching mathematician "must provide clear solution methods to maintain trust, learners must not think that the teacher also has no solution". This view is in line with a traditional problem-solving approach. PSMT 10 seemed to suggest that the teaching mathematician has an absolute power in legitimizing what is to be learned. This view of teaching aligns with a traditional conception of problem solving. PSMT 10's inconsistent conception is not unusual. Beswick (2005a) noted that newly employed teaching mathematicians tend to have mixed conceptions about the teaching and learning of mathematics, whereas experienced teaching mathematicians appear to have stable conceptions.

Supporting PSMT 10's initial view, PSMT 15 disagreed with item 17, saying that the role of the teacher is to facilitate learning and the role of learning mathematicians is to find the solution to the problems. Similarly, learning mathematicians should learn under the auspices of the teaching mathematician.


It is the teacher's responsibility to provide children with clear and concise solution methods for mathematical problems.
Disagree, let's the learners responsibility to find solutions for the problems & teacher will provide assistance and guidance for learners to get the correct or clear solutions PSMT 15

Despite their disagreement on item 17, the PSMTs unanimously agreed with items 18 –20, with only two PSMTs disapproving of item 21.

Item 18 stipulated that there is a fixed amount of content to be covered per grade, while item 19 advocated a sequential presentation of content. In support of the sequential and content focused instruction within the South African schooling system, PSMT 17 argued that:


Agree
→ There is syllabus that is set ~~big~~ by the government with the given time that each topic or sub-headings need to covered at school.

PSMT 17 view is not surprising. In South Africa, the school curriculum is exam driven and schools are constantly monitored on content coverage. Hence, the PSMTs could be drawing on their school experience as learning mathematicians or prospective teachers when responding to these items. PSMT 17's view on curriculum seems to support what has been termed anti-didactical instruction (Gravenmejier, 1999). This means that the end products of others' mathematisation is broken down into sequential units of instruction, that are then presented to learning mathematicians. Additionally, PSMT 17 assumed that learning mathematicians are passive absorbers of already made knowledge. PSMT 17 further placed the teaching mathematician at the centre of teaching and assumed that a teacher's instruction is adequate for the attainment of learning outcomes. It is also essential to note that all PSMTs, except PSMT 5 and 20, agreed that mathematics means computations. Hence, it may be concluded that these cohort of PSMTs seem to have a traditional view of mathematics teaching.

For PSMT 5, mathematics is more than computations in that it requires one to reason and draw conclusions based on the computation, while PSMT 20 viewed mathematics as solving problems and devising strategies to solve problems.

Since the majority of PSMTs' undisputedly agreed with items 18 – 21, this researcher can only infer that there was a traditional conception of problem solving in this cohort of PSMT. Item 20 clearly indicated an expository instructional strategy which is in stark contrast to a progressive conception of problem solving.

6.3.4 Traditional conception of problem-solving pedagogy

Items 22 – 27 sought to determine the PSMTs' views about the teaching mathematician's role in teaching. These items were concerned more with traditional ways of teaching mathematics. The PSMTs refuted almost all the items, except item 25, which split PSMTs into two opposing groups of four (4) and three (3). The PSMTs who disapproved of these items confirmed their earlier stance for progressive pedagogy in teaching mathematics. This implies that PSMTs do not see themselves as providers of ready-made knowledge but facilitators and participants in the knowledge construction process. In support, PSMTs 5, 7 and 8, respectively, responded to item 23 as follows:

Disagree – a life long learner that will show ~~meet that me that learner~~ I have to see where did I go wrong or see how the learner came up with an answer PSMT 5

~~I Agree~~ I disagree
I will be happy if the Learner came up with the solution. PSMT 7

PSMT 8
No I disagree I donot know everything there are thing I will learn from the learners and help he to grow so it is good to have learners whoare confident enough to raise their point of view

The views of PSMTs 5, 7 and 8 seem to align with a progressive problem-solving agenda in mathematics teaching. They gave learning mathematicians a leading role in learning and positioned teaching mathematicians as facilitators and learning mathematicians in the process. Their views are supported by the Department of Basic Education (2018, p.18), which argued that:

> *Learners need to develop their ability to think out of the box (i.e. to find strategies that have not been shown to them before) since this is useful for effective problem solving in mathematics. Bearing this in mind, the framework provides an example per phase of a more ‘open’ problem in addition to the more standard curriculum linked examples. When learners ‘think out of the box’ they are applying mathematical reasoning.*

It may be concluded that learning mathematics in a South African context is no longer about ‘stock piling’ knowledge, but rather about what learning mathematicians can do with their knowledge.

In summary, this cohort of PSMTs appears to have a mixed conception of problem solving. This could be an indication that PSMTs are still developing their own conceptions of problem solving, based on their learning of mathematics at school and ITE programmes. This dual conception could also be the result of curriculum pressures experienced by PSMTs, as most of them were teaching Grade 12s on Saturdays and during their university shut down, to assist high school learning mathematicians in improving their pass rates.

6.4 Summary

In solving non-routine problems in financial mathematics, the PSMTs drew mostly on algebraic, numerical, verbal and pictorial strategies. This is not surprising since most LTSMs and textbooks in South Africa teach FM through these strategies. The absence of graphical strategy is alarming since graphs have become the mostly used mode of communication due to their ability to compress information and be eye-catching. In using these strategies, PSMTs employed transformational, representational, linguistic and contextual activities of algebra. Justifying and generalizing activities of algebra did not appear in the PSMTs’ work (Kilpatrick et al. (2001). This absence may be attributed to the nature of the non-routine problems posed in this study, rather than the PSMTs’ inability to recognize and apply them. As indicated earlier, the problems were open to being solved using multiple strategies but they were close-ended, in that they had a fixed answer. Further, the non-routine problems did not require PSMTs to prove or establish any mathematical fact.

In grappling with algebraic activities, the PSMTs confronted difficulties which led them into committing linguistic and contextual errors more than mathematical errors. Contextual errors stemmed from insufficient understanding of financial principles, such as TVM and compounding of interest, while linguistic errors emanated from a misinterpretation of the text or an inappropriate application of definitions, especially in calculating periodic interest rate.

Despite their inability to define or explain the term ‘mathematical problem’ item 3 on the questionnaire (see appendix I), PSMTs who took part in this study seemed to align themselves with a progressive problem-solving pedagogy. They argued that a knowledge of various problem-solving strategies would greatly help learning mathematicians to choose appropriate strategies when solving problems. However, this cohort of PSMTs appeared to have a mixed conception of problem solving, which means that they subscribed to both a traditional problem-solving conception and a progressive problem-solving conception.

This researcher concludes that this cohort of PSMTs have an extensive knowledge of mathematics and problem-solving strategies, which they executed efficiently when required. However, their use of this extensive knowledge seems to be inflexible, since the PSMTs were not able to adapt this knowledge to the financial context. As a result, they failed to successfully solve non-routine mathematical problems in financial mathematics. This could be attributed to the PSMTs’ inability to deal with mathematical problems, which implicitly demanded the use of the TVM principle. It may also be attributed to the PSMTs’ inability to make appropriate assumptions about the problem variables, their failure to identify variables needed to resolve the problem and their strategy rigidity.

CHAPTER SEVEN: SUMMARY OF STUDY, IMPLICATIONS AND RECOMMENDATIONS

7.1 Introduction

This chapter provides an overview for the previous chapters and briefly summaries the findings from chapter six. Based on the findings, recommendations have been made. This study has been concluded by stating its limitations.

In chapter one, this researcher put forth the rationale for this study and his assumptions regarding the problem-solving abilities of PSMTs, based on his many years of teaching at various high schools and in ITE programmes. Of significance in the context of this research were the PSMTs' proficiency in problem-solving strategies and the extent to which such proficiencies influenced their solution processes. While this significance was to some extent supported by other researchers, this study shows that this cohort of PSMTs possessed a wealth of problem-solving strategies and algebraic activities. The PSMTs' usage of this knowledge in solving non-routine problems in financial mathematics seemed to have been hampered by their insufficient knowledge of the time value of money principle and contextual activities of algebra.

The following questions were posed as gauges for the exploration of PSMTs problem-solving strategies, algebraic activities and their conceptions of mathematical problem solving:

1. What are the cognitive strategies that pre-service mathematics teachers use in the context of their problem solving in financial mathematics?
2. To what extent do pre-service teachers exhibit intra-task or inter-task strategy flexibility?
3. What are pre-service mathematics teachers espoused conceptions concerning non-routine mathematical problem solving?
4. To what extent do pre-service teachers draw on algebraic activities when solving non-routine mathematical problems?

To find answers to these questions, this researcher examined which teaching approach was advocated by the South African School curriculum in mathematics (see chapter one). The CAPS document revealed that while teaching mathematics through problem solving is the advocated approach, it is not prioritized in assessment.

Chapter three examined the problem-solving strategies which are used in solving non-routine problems (Krulick and Rudnock 1982; Posamentier and Krulik 2008), as well as the algebraic activities with which problem solvers grapple during mathematical problem solving (Kilpatrick et al., 2001). The importance of financial mathematics within the school curriculum and beyond was highlighted together with the challenges of solving problems in financial mathematics, as an applied branch of mathematics.

In chapter four, the theoretical underpinnings of the study were discussed and the analytical constructs which were used to explore PSMTs' problem-solving abilities in financial mathematics were developed.

In chapter five, this researcher discussed the research paradigm, research approach and the design of the study. The context of the study, the participants, the instrument and the methods of data generation and analysis were expanded upon.

7.2 Summary of the Study

7.2.1 What are the cognitive strategies that pre-service mathematics teachers use in the context of their problem solving in financial mathematics?

In chapter six, the findings of this study were advanced. This study found that PSMTs were able to use the following cognitive problem-solving strategies flexibly:

1. Numerical strategy.
2. Algebraic strategy.
3. Verbal strategy.
4. Pictorial strategy.

None of the groups used a **graphical strategy** in solving the problems, even though problem 4 required the use of this strategy. This is worrying since graphs are the most economical mode of mathematical communication, even used in news bulletins and advertising. The absence of this strategy in the PSMTs' repertoire could be an indication of the PSMTs' strategy rigidity or even a lack of knowledge in communicating their mathematical ideas graphically.

7.2.2 To what extent do pre-service teachers draw on algebraic activities when solving non-routine mathematical problems?

It was noted that no group engaged in the **generalizing and justifying activities** of algebra. The absence of generalizing and justifying activities in the PSMTs' work could be attributed to the kind of non – routine mathematical problems posed in this study, rather than an indication that PSMTs have developed an inert knowledge of mathematics. None of the problems posed to the PSMTs directly required the used of generalizing and justifying activities i.e. none of the non – routine mathematical problems required PSMTs to prove or investigate any mathematical fact. However, PSMTs grappled with **representational, transformational, linguistic** and **contextual** activities of algebra in their search for solutions to problems in financial mathematics. While the PSMTs made fewer mathematical errors, there were more contextual and linguistics errors in their solution processes. For example, the D_group ignored the TVM principle in almost all the problems they answered (one should bear in mind that these prospective teachers were not taking the FM module, although it was their last semester at the institution) and like the other groups had problems with the technical usage of words in FM problems. The D_group seemed to have been restricted to using numerical strategy across all problems. This could be an indication that they have developed inert knowledge of financial mathematics concepts and problem-solving strategies (Boaler, 2002; Da Silva et al., 2009).

As indicated earlier, the problems posed in this study were open to being solved using a variety of problem strategies but all the problems had a fixed answer. As a result, PSMTs did not grapple with the generalizing and justifying activities, although the PSMTs had to make judgments with regard to the timing of the initial deposit (see Problem 2) and the most appropriate time (day) for borrowing money from the bank (see Problem 1). The PSMTs were able to vary their problem-solving strategies flexibly according to the demands of the problem although they could not always change to a more efficient problem-solving strategy. By including the contextual activities of algebra, this study has supported and extended the findings of Da Silva et al. (2009), by showing that limited knowledge of financial principles, such as TVM and compounding of interest, led the PSMTs into committing contextual errors when solving problems in FM.

7.2.3 To what extent do pre-service teachers exhibit intra-task or inter-task strategy flexibility?

The study has further shown that mathematical proficiency in general and algebraic activities in particular, as described by Kilpatrick et al. (2001), is inadequate as a measure of PSMTs' problem solving proficiency, since PSMTs who demonstrated capability and flexibility to switch problem solving strategies and algebraic activities while solving a problem or problems could not attain the expected results due to their limited contextual knowledge of financial mathematics. Hence, there was need to extend algebraic activities to incorporate contextual activities necessary for solving non-routine problems in applied or contextualized problems, such as those found in financial mathematics or everyday contexts. Further one noted that PSMTs' switching of problem – solving strategies was either the result of PSMTs' hitting a dead end or it was dictated the nature of the problem.

7.2.4 What are pre-service mathematics teachers espoused conceptions concerning non-routine mathematical problem solving?

The PSMTs who participated in this study seemed to have a hybrid (mixed) conception of problem solving. They seemed to advocate teaching mathematics through problem solving but actually subscribed to traditional and progressive pedagogies on problem solving. Beswick (2005a) reported the same phenomenon about newly qualified teachers who partook in his study. As indicated earlier, the PSMTs' hybrid conception of mathematical problem-solving and pedagogy could be a result of how they have come to know mathematics or a result of an external voice such peers, colleagues, learning mathematicians, etc. (Beswick, 2005a; Ernest, 1989).

7.3 Implications for ITE Programmes, Teacher Development and Interventions

While the findings of this study limit generalizations, they have provided sufficient background and understanding from which cautious recommendations for ITE programmes, mathematics teacher development and in-service interventions may be drawn.

The analysis has revealed that PSMTs had minimal challenges with mathematics. In most cases, PSMTs' solution processes were procedurally accurate but lacked contextual awareness when solving financial mathematics' problems, which resulted in incorrect answers. Additionally, the PSMTs lacked the necessary language to communicate their mathematics as they could not appropriately define the variables they were using and some referred to a timeline as a number line.

ITE programmes and teacher educators (i.e. course instructors/lecturers/ professors) should set, implement and assess learning outcomes that do not only deal with the content of financial mathematics but also the language through which the content is relayed. Financial mathematics module(s) should be made compulsory in this institution because it is a critical component of school mathematics. Further, the FM module is critical for the consumption of financial products, such as the purchasing and financing of a home, a car, planning for retirement and choosing an investment option (Gardner, 2004) in adulthood. This module is the only source of financial literacy skills in SA. Hence it is intended to play a key role in curbing the ever-increasing levels of indebtedness among the citizens and attaining financial security.

In teaching FM, application, of the TVM principle and the effect of compound interest on money, should be the focus of the module, instead of the mere academic usage of mathematical procedures (e.g. substitution of values into a formula).

Further, in teaching this topic, the account balance approach (AB) should precede the individual payment approach (IP) (Pournara, 2013a), as it draws on learning mathematicians' intuitive knowledge and problem-solving strategies.

It is essential to emphasize the use of timelines when solving multiple cash flow problems and to relate this to a geometric series. Using timelines would depict the timing of the deposit and thus enabling learning mathematicians to decide on whether all cash flows earn interest or not. Using a geometric series as a starting point in solving multiple cash flow problems, instead of a formula, would afford teaching mathematicians and instructors an opportunity to focus on important financial principles, such as TVM and interest calculations, instead of whether the payment/deposit was made at the beginning or at the end.

As shown by the solutions of the T_group and the M_group to problem 2, using a formula may be superficial because in a well-structured problem, the number of unknowns in the formula may correspond to the given values in the problem statement, leaving the required unknown without a value. This may lead unsuspecting problem solvers to think that all is well, but it results in incorrect answers.

ITE instructors/lecturers should adopt teaching financial mathematics through problem solving, instead of teaching for problem solving, as this might give them a room to address linguistic issues. It is also necessary for the institution to develop a professional development module on financial mathematics (and financial literacy) which in-service teachers could take to advance their knowledge of finance, to enhance their teaching of financial mathematics.

The Department of Basic Education needs to run developmental intervention workshops, particularly for newly qualified teaching mathematicians, which address not only the content of financial mathematics, but also the technical language that accompanies financial mathematics.

The absence of any graphical mode of communication in PSMTs' solutions should not be ignored. Kolovou et al. (2011) expounded that newly taught problem-solving strategies turn to overshadow problem solvers' intuitive strategies. It is therefore essential that curriculum developers, implementers and LTSM developers, particularly textbooks authors, give equivalent space and attention to all modes of communicating mathematics, as advocated by CAPS. In this way, PSMTs would develop adequate knowledge of graphical strategy.

7.4 Recommendations

Although the findings of this study have been based on a small sample, recommendations are made on educational policy, teacher education instittutions and further research.

7.4.1 Recommendation on educational policy

- LTSMs used in schools teach this topic through the use of a formula, which leaves learning mathematicians unprepared for solving non-routine problems frequently encountered in workplaces or out-of-school contexts. It is recommended that the use of account balance approach should be emphasized in learning materials
- Make the time value of money principle visible by including problems that require PSMTs and learning mathematicians to work out the duration of a loan or investment prior to calculating loans or repayments in LTSMs
- Make financial literacy (currently offered as part of FM) an independent and compulsory module in initial teacher education. The PSMTs' inability to make financially sound decisions about their computations and failure to get the expected answers indicates their lack of financial literacy, rather than their lack of mathematical knowledge and skills, since, in their work, once the decision was made, all the procedures were expertly executed.

7.4.2 Recommendation for teacher education institutions

- Teachers should use the account balance approach as a starting point in teaching financial mathematics across all grades and phases. Most LTSMs used in schools teach this topic through the use of a formula, which leaves learning mathematicians unprepared for solving non-routine problems frequently encountered in workplaces or out-of-school contexts
- Make the time value of money principle visible by including problems that require PSMTs and learning mathematicians to work out the duration of a loan or investment prior to calculating loans or repayments.
- Minimize rushing PSMTs into vertical mathematization but focus on integrating their intuitive problem-solving strategies.

- Emphasize the use of representations, such as timelines, when dealing with multiple cash flows.
- Develop and monitor learning outcomes which cover language aspects, particularly on finance.
- Make error analysis part of PSMT learning programmes (Khalo & Bayaga, 2014). This could assist them in identifying not only their own errors but those that are likely to be made by others.
- Provide PSMTs with opportunities to analyse problems from different points of view and to systematically evaluate complex situations.

7.4.3 Recommendations for further research

- Include paticipants from more than one programme and more institutions to provide a richer view of the problem-solving abilities of prospective teachers as they exit their initial teacher education programmes.
- Compare the problem solving strategies use by PSMTs doing FM in affluent institutions to the strategies used by PSMTs doing FM in rural institutions.
- Investigate the problem-solving strategies and algebraic activities of the PSMTs from previleged communities in South Africa.

7.5 Limitations of the Study

In using the finding of this study, one should be aware that the study was conducted in an institution of higher learning in South Africa, which is in a rural area, with low entry requirements into the ITE programme. The student population in this institution is mainly black South Africans, from disadvantaged communities. Hence the problem-solving strategies, algebraic activities and problem-solving conceptions reported here were of less privileged South Africans. The prospective teachers were Grade 12 graduates mainly from schools labelled as ‘underperforming schools’. The participants were not a representative sample of students in South African higher education or of students in rurally-based institutions in South Africa but a unique case, illuminating the problem-solving experiences of the mostly disadvantaged communities.

A qualitative research design was adopted, which thus limited statistical generalizations. However, a single case study with multiple units was used to enable intra and inter group comparisons. Participants were drawn from prospective teachers taking at least one mathematics module in their final year. Hence, the findings may not be extrapolated beyond this cohort of participants. However, the constructs, such as verbal strategies, graphical strategies, contextual activities etc., generated from this study were not only developed from the data generated in this study but from existing literature concerning mathematics education and South African policy documents on mathematics education, such as CAPS, and consequently these construct could be used beyond this study to describe PSMTs' learning experiences. Only 11 participants from 30 invited took part in the problem solving sessions. This limited the number of groups to be studied which could have impacted the findings.

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APPENDICES

Appendix A: Letter to the Dean of the Faculty of Education

University of Zululand

P/Bag X1001

KwaDlangezwa

3886

12th March 11, 2019

The Dean

Faculty of Education

University of Zululand

P/Bag X1001

KwaDlangezwa

3886

Dear Sir

Re: Permission to conduct a research study at your institution

I am currently a registered student at your institution studying towards Doctor of Education in mathematics Education. I am writing a thesis on **An Exploration of Final Year Pre-Service Mathematics Teachers Learning Experiences in Solving Non-Routine Financial Mathematics Problems at a South African University**. I would like to conduct this research at your institution involving your level 4 students majoring in mathematics. The data collection process will take one week.

I will not interrupt or interfere with the normal activities of the institution. Data collection process will take place after 16h30 each day. All students will be informed that this activity is not part of their formal academic programme and will not earn any credits by participating. However, I will ask them to voluntarily partake in the study. Students who volunteer will work in groups to solve the given problems. I expect the students to spend 15 – 45 minutes in solving a problem and they will be given one problem per day.

Full anonymity and confidentiality are assured. In compiling the thesis, reporting its findings the names of the participants and the institution will not be used. I am intending to observe, audio and video record the students as they solve the problems. I further intend to interview some groups. I all the participants will also respond to a questionnaire on conceptions about mathematics, its teaching and learning.

I therefore request your permission to involve these students in this study and also to use the university premises.

Your assistance is highly valued

Kind regards

Student name: Talasi, T.   Student signature

Student number: 201760013

Dr Krishnannair, A.

Supervisor   Supervisor's Signature

Appendix B: Letter to the Head of Department

P/Bag X1001

KwaDdlangezwa

3886

12th March 2019

The Head of Department

MSTE Department

Faculty of Education

Dear Madam

Research title: An Exploration of Final Year Pre-Service Mathematics Teachers' Learning Experiences in Solving Non-Routine Financial Mathematics Problems at a South African University.

I, T. Talasi, am currently enrolled for a Doctor of Education in mathematics education at the University of Zululand under the supervision of Dr Krishnannair. I request permission to involve our level 4 students majoring in mathematics in my study.

The purpose of this study is to explore pre-service mathematics Teachers' (PSMTs') learning experiences with a special emphasis on the mathematical problem-solving strategies used by this cohort of mathematics teachers as they solve non-routine mathematical problems. I plan to work with those who consent over a one-week period. During this period, I will observe, audio and video record the students as they solve mathematical problems. The students will be working in groups during this period. Some groups will be interviewed.

The study is important because it endeavours to describe the learning experiences of pre-service mathematics Teachers (PSMTs) in solving non-routine financial mathematics problems. It will also shed some light into PSMTs' conceptions of mathematics, its teaching and learning. This study would be of great help to curriculum developers, policy designers, textbook authors, subject advisors and lecturers as they are likely to know the extent to which these teachers would embrace problem solving in their teaching. The study will further shed light into how the researcher could improve his teaching approaches and what curriculum changes are needed in the training future cohorts of mathematics teachers to meet the local and global standards.

All information collected will be kept confidential and will be used for research purposes only. Neither the names nor personal details (labels) of the participants or the institution will be revealed in reporting the findings of this study. Participants would be free to withdraw from the study at any time and where possible information collected on such participants will not be used in compilation of the thesis and on reporting its findings.

I, hoping that my request will meet your favourable attention, thank you in advance.

Yours faithfully

Talasi, Tatolo

Signature: _____________

Telephone: 035 902 6064

Supervisor:

Dr A. Krishnannair

Signature: _____________

Telephone: (035) 902 6865

Appendix C: Letter to PSMTs

P/Bag X1001

KwaDlangezwa

3886

12th September, 2019

Dear Student

I, Tatolo Talasi, hereby request you to voluntarily partake in my research. I am currently enrolled for a Doctor of Education in mathematics education at the University of Zululand under the supervision of Dr Krishnannair. The title of my thesis is:

> ***An Exploration of Final Year Pre-Service Mathematics Teachers' Learning Experiences in Solving Non-Routine Financial Mathematics Problems at a South African university***.

The purpose of this study is to explore pre-service mathematics Teachers' problem-solving strategies as they solve non-routine financial mathematics problems. I plan to work with the volunteers over a one-week period. The volunteers will be asked to solve problems in groups of five or less. One problem will be solved per day and each session is expected to take between 15 and 45 minutes. During each session, I will observe, audio and video record the groups as they solve mathematical problems. Furthermore, I will interview some groups based on how they solved the problem. All the information collected will be used for research purposes only and no names or other details of the participants or institution will be used in compiling the thesis or reporting the findings of this study.

Please note that

1. Activities will take place after 16hrs30 each day
2. Activities done will not form part of your assessment for any module
3. You may withdraw from the study at any time without informing me and there will be no consequences
4. You may choose not participate. There is no penalty for not participating as there are no rewards for participating.

If you choose to participate, please complete and sign the attached consent form. The consent form should be returned to me.

Should you require more information, please feel free to contact me or my supervisor by email or calling.

Kind regards

Talasi, Tatolo

Cell: 035 902 6064

Supervisor

Dr Krishnannair

Telephone: (035) 902 6865

Appendix D: Consent Form for Students

Research title:

An Exploration of Final Year Pre-Service Mathematics Teachers' Learning Experiences in Solving Non-Routine Financial Mathematics Problems at a South African University

I (name of student) ………………………….. with UNIZULU student number ………………….. hereby agree to participate in the study conducted by Mr. T Talasi, under the supervision of Dr Krishnannair. I understand that the purpose of the study is to explore pre-service mathematics teachers' (PSMTs') learning experiences with a special emphasis on the mathematical problem-solving strategies used by this cohort of mathematics teachers as they solve non-routine financial mathematics problems.

Please tick the most appropriate

	Yes	No
1. Do you voluntarily agree to participate in this study?	[ ]	[ ]
2. Do agree that participation should be voluntary?	[ ]	[ ]
3. Do you agree that you may withdraw at any time?	[ ]	[ ]
4. Do understand that no marks to be awarded for participation?	[ ]	[ ]
5. Do you understand that activities done in this study do not form part of any module?	[ ]	[ ]

6. Do you understand that no grades/marks will be awarded for participation?

[ ] [ ]

7. Do you agree that information should be used for research purposes only?

[ ] [ ]

8. Do you agree to be voice recorded as a group?

[ ] [ ]

9. Do you agree to be video recorded as a group?

[ ] [ ]

10. Do you agree that your solutions of the problems will be taken by the researcher?

[ ] [ ]

Signature of student ……………………..

Date …………………………………………

Appendix E: Ethical Clearance Certificate

UNIVERSITY OF ZULULAND
RESEARCH ETHICS COMMITTEE
(Reg No: UZREC 171110-030)

RESEARCH & INNOVATION
Website: http://www.unizulu.ac.za
Private Bag X1001
KwaDlangezwa 3886
Tel: 035 902 6731
Fax: 035 902 6222
Email: DlaminiA@unizulu.ac.za

ETHICAL CLEARANCE CERTIFICATE

<table>
<tr><td>Certificate Number</td><td colspan="8">UZREC 171110-030 PGD 2018/244</td></tr>
<tr><td>Project Title</td><td colspan="8">AN EXPLORATION OF FINAL YEAR PRE-SERVICE MATHEMATICS TEACHERS' LEARNING EXPERIENCES IN SOLVING NON-ROUTINE FINANCIAL MATHEMATICS PROBLEMS AT A SOUTH AFRICA UNIVERSITY</td></tr>
<tr><td>Principal Researcher/ Investigator</td><td colspan="8">T Talasi</td></tr>
<tr><td>Supervisor and Co-supervisor</td><td colspan="4">Dr A Krishnannair</td><td colspan="4"></td></tr>
<tr><td>Department</td><td colspan="8">Mathematics, Science and Technology Education</td></tr>
<tr><td>Faculty</td><td colspan="8">Education</td></tr>
<tr><td>Type of Risk</td><td colspan="8">Med Risk– Data collection from people</td></tr>
<tr><td>Nature of Project</td><td>Honours/4th Year</td><td></td><td>Master's</td><td></td><td>Doctoral</td><td>x</td><td>Departmental</td><td></td></tr>
</table>

The University of Zululand's Research Ethics Committee (UZREC) hereby gives ethical approval in respect of the undertakings contained in the above-mentioned project. The Researcher may therefore commence with data collection as from the date of this Certificate, using the certificate number indicated above.

Special conditions:

(1) This certificate is valid for 1 year from the date of issue.
(2) Principal researcher must provide an annual report to the UZREC in the prescribed format [due date-04 October 2019]
(3) Principal researcher must submit a report at the end of project in respect of ethical compliance.
(4) The UZREC must be informed immediately of any material change in the conditions or undertakings mentioned in the documents that were presented to the meeting.

The UZREC wishes the researcher well in conducting research.

Professor Gideon De Wet
Chairperson: University Research Ethics Committee
Deputy Vice-Chancellor: Research & Innovation
04 October 2018

CHAIRPERSON
UNIVERSITY OF ZULULAND RESEARCH
ETHICS COMMITTEE (UZREC)
REG NO: UZREC 171110-30
05 -10- 2018
RESEARCH & INNOVATION OFFICE

Appendix F: Turnitin Report

TALASI_TURNITIN_DOCUMENT

ORIGINALITY REPORT

7%	4%	4%	%
SIMILARITY INDEX	INTERNET SOURCES	PUBLICATIONS	STUDENT PAPERS

PRIMARY SOURCES

1	Abdullah, Abdul Halim, Nor Hasniza Ibrahim, Johari Surif, Marlina Ali, and Mohd Hilmi Hamzah. "Non-routine mathematical problems among in-service and pre-service mathematics teachers", 2014 IEEE International Conference on Teaching Assessment and Learning for Engineering (TALE), 2014. Publication	<1%
2	uir.unisa.ac.za Internet Source	<1%
3	journals.upd.edu.ph Internet Source	<1%
4	"Investigating the Development of Pre-Service Teachers' Problem-Solving Strategies via Problem-Solving Mathematics Classes", European Journal of Educational Research, 2020 Publication	<1%
5	scholar.sun.ac.za Internet Source	<1%

6	hdl.handle.net Internet Source	<1%
7	www.amesa.org.za Internet Source	<1%
8	www.bridge.org.za Internet Source	<1%
9	"Mathematics Curriculum in School Education", Springer Science and Business Media LLC, 2014 Publication	<1%
10	"Broadening the Scope of Research on Mathematical Problem Solving", Springer Science and Business Media LLC, 2018 Publication	<1%

Appendix G: Editor's Report

Helen Bond

IMPELA EDITING SERVICES

impelaediting@gmail.com

079 395 5873

06 August 2020

CERTIFICATE

Tatolo Talasi

TalasiT@unizulu.ac.za

Dear Tatolo

Thank you for using Impela Editing Services to proofread your PhD thesis entitled, "*An Exploration of Final Year, Pre-Service Mathematics Teachers' Learning Experiences in Solving Non-Routine Financial Mathematics Problems at a South African University*".

We have proofread for errors of grammar, punctuation, spelling, syntax and typing mistakes. We have formatted your work according to APA guidelines and checked the references (this means checking the formatting).

Please note that Impela Editing does not accept any fault for changes made to a document after emailing the final draft and issuing a certificate.

I wish you the very best in your submission.

Kind regards

Helen Bond (Bachelor of Arts, HDE)

Appendix H: Non-Routine Mathematical Problems

Research questions:	
8	**To what extend are pre-service teachers exhibiting intra-task or inter-task strategy flexibility?**
9	**To what extend do pre-service teachers draw on algebraic activities when solving non-routine mathematical problems?**

a) Do not write your name(s).
b) Show all the working clearly.
c) Use more than one method in answering the question.

1. An amount of R1 000 is invested at an annual interest rate of 6%. Interest is compounded **every three months** (quarterly). After 5 years the investment in rands, will be worth…..
2. A merchant receives an invoice worth R2800 payable in 30 days. The invoice states that if payment is made in 10 days, there will be a 3% p.a. simple discount. If the payment is not made in thirty days, there will be a R30 delinquency charge for every thirty days thereof.
   Would it be worthwhile for him to borrow from a bank that charges interest at 5% p.a. simple interest in order to take advantage of the discount? Justify your answer mathematically?

   (Adapted May, 1951).

3. You open a NedTerm account with Nedbank in April 2008. This account allows you to make monthly deposits. When you open the account you make an initial deposit of at least R1000.00 and then monthly payments of R100 or more. You decided to deposit initial amount of R1000.00 and monthly deposit of R300. These payments are made at the start of every month and you do this for 18 months. The monthly payments start in May 2008. The interest rate is 7.65% p.a. compounded monthly. How much money will you accumulate over this period?

   (Adopted from Pournara, 2015)

4. A debt of R200 000 bearing interest at 5% p.a. compounded semiannually will be retired by R50 000 payment in 2 years plus a sequence of 8 equal semiannual payments starting at the end of 3rd year. Find the regular payment.
   (Adapted May, 1951).
   A man, aged 25 pays R5 000 at the end of year for the next 35 years to an investment company, he will receive R300 000 at the age of 60. At what annual rate is the interest being credited? (Adapted May, 1951).

Appendix I: Group Solutions

T_group: Solutions

Problem 1:

1. Discount for paying in 10 days

R 2800 X $\frac{3}{100}$ ~~= R 2716,00~~

= R 84,00

∴ Discount will be R 84;00

∴ He/she must borrow from the bank

R 2800 – R 84

= R 2716,00

He/sher will borrow R 2716,00 from the bank. to take advantage of the discount.

If he/she does not go to the bank there will be R30 charge every 30 days.

If he/she pays after the first 30 days he will have to pay:

R 2800 + R30

= R2830.

If he/she borrows from the bank.

$A = P(1 + in)$

$A = R2716\left[1 + \frac{5}{100}\left(\frac{1}{12}\right)\right]$

$A = R\,2727,32$

∴ R2727,32 − R2716 = R11,32

At the bank there will be a charge of R11,32 after every thirty days thereof.

Then R30 − R11,32 = R18,68

∴ R18,68 will be saved if he/she goes to borrow the money from the bank.

~~Payment~~

If he/she pay the money ~~merchant~~ after 30 days without taking the loan from the bank. The payment will be R 2830,00

~~But if he/she pay the merchant~~.

If he/she borrows from the bank.

$A = P(1 + in)$

$A = R2716\left[1 + \frac{5}{100}\left(\frac{1}{12}\right)\right]$

$A = R\,2727,32$

∴ R2727,32 − R2716 = R11,32

At the bank there will be a charge of R11,32 after every thirty days thereof.

If he/she borrows the money from the bank and pay the bank after 30 days he/she would be required to pay R 2727,32

∴ R 2830,00 − R 2727,32
= R102,68

∴ He/she would save R 102,68

∴ It will be worthwhile for him to borrow from a bank charges interest at 5% p.a

Problem 2

Initial deposit = R1000.00

For the period of (1 + 18) months = 19 months

$$A = P(1+i)^n$$

~~A = P(1 +~~

$$A = R1000\left(1+\frac{7.65\%}{12}\right)^{\frac{19}{12}\times 12}$$

$$= R1000\left(1+\frac{7.65\%}{12}\right)^{19}$$

$$= R1128.332125$$

For the R 300. rands deposited everymonth

For 18 months.

$$Fv = \frac{x\left[(1+i)^n - 1\right]}{i}(1+i)$$

$$= \frac{300\left[\left(1+\frac{7.65\%}{12}\right)^{\frac{18}{12}\times 12} - 1\right]}{\frac{7.65\%}{12}}\left(1+\frac{7.65\%}{12}\right)$$

$$Fv = R\,5739.158841$$

~~First~~

~~Accumulated money is~~

R1128.332125 + R5739.156841

= R6867.49.

∴ ~~The money accumulated~~

Deposited money & monthly payments

R1000 + R300 × 18 = R6400

$$Fv = \frac{x\left[(1+i)^n - 1\right]}{i}(1+i)$$

$$= \frac{300\left[\left(1+\frac{7.65\%}{12}\right)^{\frac{18}{12}\times 12} - 1\right]}{\frac{7.65\%}{12}}\left(1+\frac{7.65\%}{12}\right)$$

$Fv = R\,5739\ 158841$

~~First~~

~~Accumulated money is~~

R 1128.582125 + R 5739.156841

= R 6867.49.

∴ ~~The money accumulated~~

Deposited money & monthly payments

R1000 + R300 × 18 = R6400

∴ Accumulated money over 18 months period

R6867, 49 − R6400 = R467,49

∴ He/she accumulated an interese of R469,49

Problem 3: First attempt

$$FV = x\left[\frac{(1+i)^n - 1}{i}\right]$$

$$FV_1 = R5000\left[\frac{(1+i)^1 - 1}{i}\right]$$

$$FV_2 = R5000\left[\frac{(1+i)^2 - 1}{i}\right]$$

$$FV_3 = R5000\left[\frac{(1+i)^3 - 1}{i}\right]$$

$$FV_{35} = R5000\left[\frac{(1+i)^{35} - 1}{i}\right]$$

$$R30000 = R5000\left[\frac{(1+i)^{35} - 1}{i}\right]$$

$$FV_{35} = FV_1 + FV_2 + FV_3 + \ldots + FV_{34}$$

① $R300000 = R5000\left[\left(\frac{(1+i)^1 - 1}{i}\right) + \left(\frac{(1+i)^2 - 1}{i}\right) + \left(\frac{(1+i)^3 - 1}{i}\right) + \ldots\right.$

② $(1+i)R300000 = R5000(1+i)\left[\frac{1+i-1}{i} + \frac{(1+i)^2 - 1}{i} + \frac{(1+i)^3 - 1}{i} + \ldots\right.$

~~② − ①: $R30000(1+i-1) = R5000[1+i-1]\left[1 + \frac{(1+i)^2 - 1}{i} + \frac{(1+i)^3 - 1}{i} + \ldots\right.$~~

~~$R3000\,i = R5000\,i\,[1$~~

Problem 3: Second attempt

$$Fv = x\left[\frac{(1+i)^n - 1}{i}\right]$$

$$R300000 = R5000\left[\frac{(1+i)^{35} - 1}{i}\right]$$

$$\frac{R300000}{R5000} = \left[\frac{(1+i)^{35} - 1}{i}\right]$$

$$60i = (1+i)^{35} - 1$$

$$(60i+1)^{1/35} = (1+i)^{35/35}$$

$$(60i+1)^{1/35} = 1+i$$

$$\frac{1}{35}\log(60i+1) = \log(1+i)$$

$$\frac{1}{35} = \frac{\log(1+i)}{\log(60i+1)}$$

$$60^{\frac{1}{35}} \cdot i^{\frac{1}{35}} + 1^{\frac{1}{35}} = 1+i$$

$$\frac{60^{1/35} \cdot i^{1/35}}{i^{1/35}} = \frac{i}{i^{1/35}}$$

$$60^{1/35} = i^{1 - \frac{1}{35}}$$

$$i^{\frac{34}{35} \cdot \frac{35}{34}} = 60^{\frac{1}{35} \cdot \frac{35}{34}}$$

$$i = 60^{1/34}$$

$$i =$$

$$(1+i) + 59^{1/34} \cdot (i+1)$$

$$60i+1 = (1+i)^{35}$$

$$\log(60i+1) = 35\log(1+i)$$

$$\frac{\log(60i+1)}{\log(1+i)} = \frac{\log 2}{\log 2}$$

$$\log 2 = \frac{70}{2}$$

$$2\log(60i+1) = 70\log(1+i)$$

$$(60i+1)^2 = (1+i)^{70}$$

$$(60i+1)(60i+1) = (1+i)^{70}$$

$$\frac{60}{1}i + \frac{60}{60} = (1+i)^{35}$$

$$\frac{3600i + 60}{60} = (1+i)^{35}$$

$$\frac{60(60i+1)}{60} +$$

$$60i + 1 + 59 - 59 =$$

$$60(i+1) = (1+i)^{35} + 59$$

$$60 = \frac{(1+i)^{35}}{(i+1)^1} + \frac{59}{i+1}$$

$$60 = (1+i)^{34} + 59(i+1)^{-1}$$

Problem 3: Third attempt

$$FV = x\left[\frac{(1+i)^n - 1}{i}\right](1+i)$$

$$\frac{60i}{1+i} = (1+i)^n - 1$$

$$A = P(1+i)^n$$

$$= R50000\left(1+\frac{5\%}{2}\right)^2$$

$$\frac{60i}{1+i} + 1 = (1+i)^{35}$$

30%

$$\frac{1+i}{60i} + 1 = \frac{1}{(1+i)^{35}}$$

$$100 \cdot \frac{30}{100}, \ldots$$

$$\frac{1+61i}{60i} = \frac{1}{(1+i)^{35}}$$

$$200000 \cdot \frac{5}{100} \cdot \frac{1}{2} = R50\ldots$$

$$\frac{1}{60i} + \frac{61}{60} = \frac{1}{(1+i)^{35}}$$

$$1 + 61i = 60i(1+i)^{-35}$$

2500

205000 − 500000

$$200000(1+i)^n$$

$$~~200000~~\left(1+\frac{5\%}{2}\right)^1 = 205000$$

$$155000\left(1+\frac{5\%}{2}\right)^1 = 158875$$

$$108875\left(1+\frac{5\%}{2}\right) = R111596{,}875$$

$$61596{,}875\left(1+\frac{5\%}{2}\right) = R63136{,}79688$$

$$13136{,}79688\left(1+\frac{5\%}{2}\right) = 13465{,}2168$$

$$R13465{,}2168\left(1+\frac{5\%}{2}\right)^2 = R14146{,}8934$$

Problem 3: Fourth attempt

$$FV = x\left[\frac{(1+i)^n - 1}{i}\right]$$

$$R300000 = R5000\left[\frac{(1+i)^{35} - 1}{i}\right]$$

$$\frac{R300000}{R5000} = \frac{(1+i)^{35} - 1}{i}$$

$$R60 = \frac{(1+i)^{35} - 1}{i}$$

$$FV = R5000\left[\frac{\left(1+\frac{28,57}{100}\right)^{35} - 1}{\frac{28,57}{100}}\right]$$

Problem 4

2. $A = P(1+i)^n$

~~A~~

$A = R200000\left(1+\frac{5\%}{2}\right)^1$

$A = R205000.$

He/she pays R50000 : Now owes R155000

$A = R155000\left(1+\frac{5\%}{2}\right)^1$

$A = R158875,00$

He/she pays R50000. Now owes R108875

$A = R108875\left(1+\frac{5\%}{2}\right)^1$

$A = R111596,875$

He/she pays R50000. Now owes R61596,875

$A = R61596,875\left(1+\frac{5\%}{2}\right)$

$A = R63136,79688$

He/she pays R50000. Now owes R13136,79688

$A = R13136,79688\left(1+\frac{5\%}{2}\right)$

$A = R13465,2168$

Does not pay for a year in accumulated money

~~A = R 13136~~, $A = R13465,2168\left(1+\frac{5\%}{2}\right)^2$

$A = R14146,8934.$

$PV = x\left[\frac{1-(1+i)^{-n}}{i}\right]$

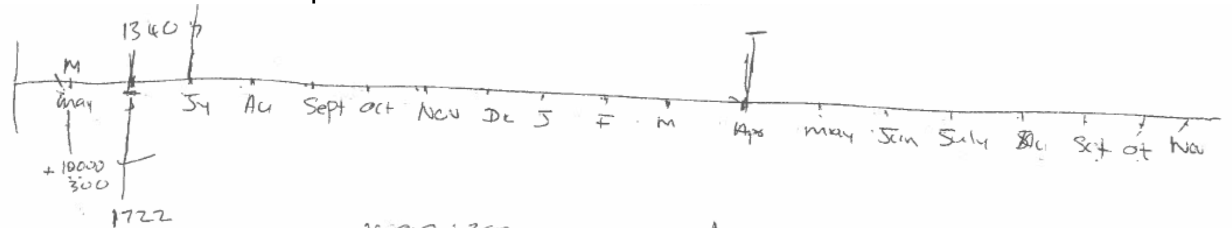
$R14146,8934 = x\,\frac{\left[1+\left(1+\frac{5\%}{2}\right)^{-8}\right]}{\frac{5\%}{2}}$

$x = R1973,029674$

$x = R1973,03.$

D_group Solutions

Problem 2: First attempt



1000+300

1300+300

$1600(1+\frac{7.65}{100})$

$= ~~13840~~$

1722.4

$A=$

$A = P(1+i)^n$

$= 1000(1+$

$2022.4(1+0.0765) = 2177.1136+300$

$2477.1136(1+0.076)$

$A = P(1+i)^n$

$= 1300(1+0.076)^{12}$

$= 3131.054364$

$= 5339.63$

.558466

$1600(1+0.0765) = ~~1399.45~~ \; 1722.4 + 300$

$2022.4(1+0.0765) =$

$1600(0.0765) = 122.4 + 1600$

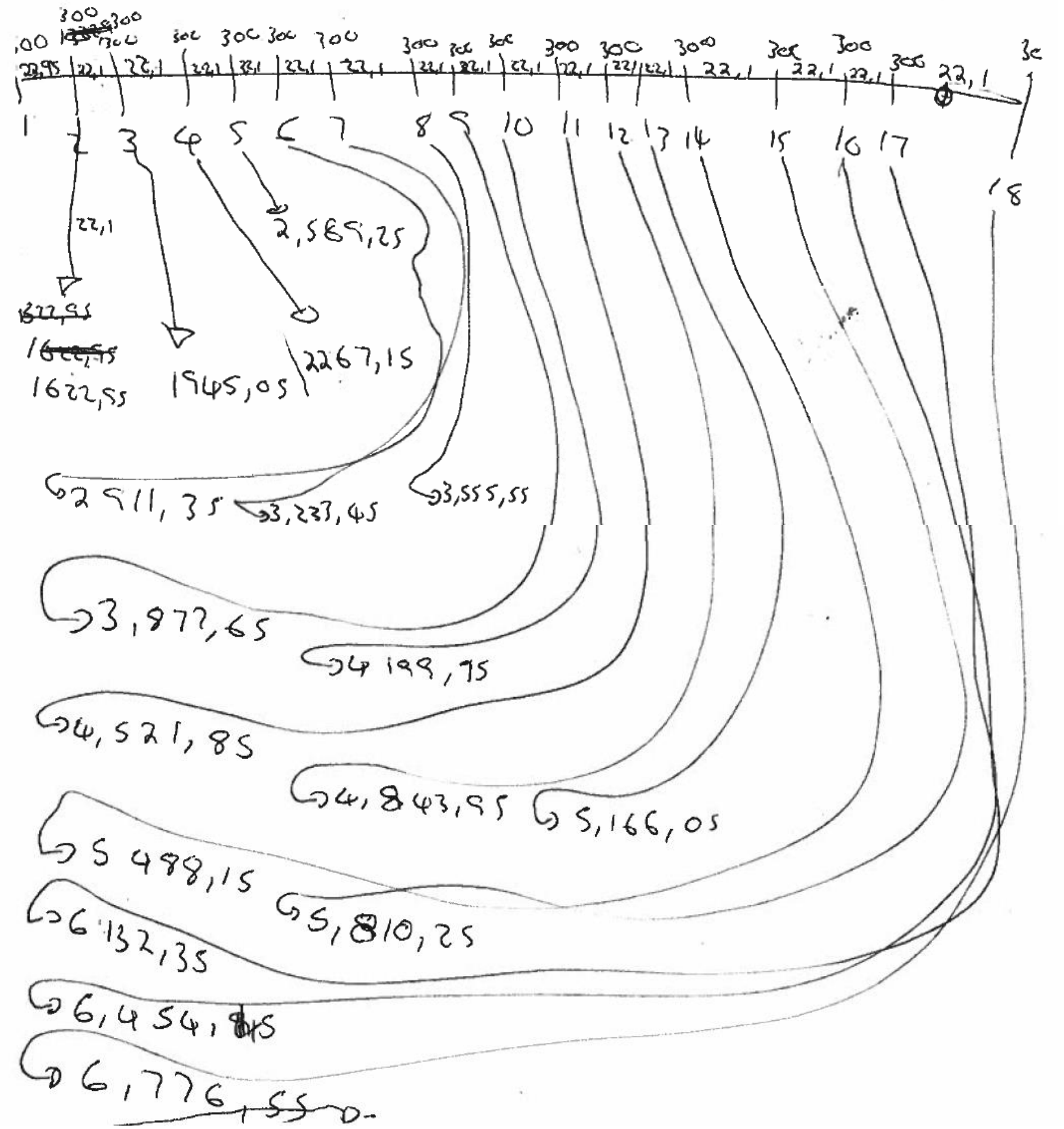
$= 1722.4$

$722.4 + 300 = 2022.4$

$2022.4(0.0765) = 154.7136 + 2022.4$

$20378.7136 =$

Problem 2: Second attempt



Problem 2: Third attempt

first month 1600(0.0765) = 122.4 interest
= 1600 + 122.4 = 2824 ✓

2824 + 300 =
3124(0.0765) = ~~238.986~~ 238.986
= 3124 + 238.986
second month ~~336279~~ = 3362.986 ✓

3362.986 + 300
3662.989(0.0765) = 280 interest
= 3662.989 + 280
= 3942.898 ✓
= 3942.898 + 300
third 394 = 4242.989

4242.989(0.0765) = 324.58865
= 4242.989 + 324.58865
= 4567.577659 ✓
4th months 4567.577659 + 300

~~4242.989~~(0.0765) =
4867.57759(0.0765) = 372.36969 + 4867.577
= ~~672.36969~~
5th month =5239.9473

5239.94735(0.0765) = 400.855 + 5239.94735
6th month = 5640.8033 + ~~300~~
5640.8033 + 300
5940.803322(0.0765) = 454.471
= 5940.8033 + 454.47
7th = 6395.274776
8th month
6395.274(0.0765) = 489.2385
= 6884.512
6884.51252

Problem 3

(1) 25 years + 35 years = 60 years

5000 × 35 = R 175 000 without Interest

R300 000 − 175000 = R125 000 Interest money that he get.

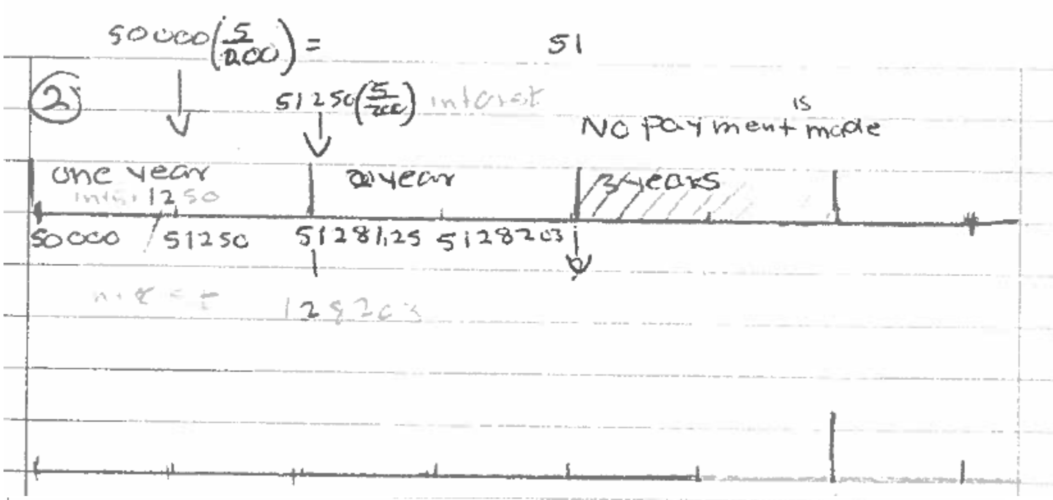
$\frac{R125\,000}{35}$ = R3 571,428571 Assuming that the money that ~~he will get on~~ It will be add into his money each year.

$5000 \times \frac{i}{100} = 3\,571,428571$

$\frac{5000i}{100} = 3\,571,428571$

$i = 71,43$

Problem 4



Semiannual = interest take place every 6 months

Debt = R 200 000.

equal payment ~~every~~ next 2 years

= 51282.03

Money needed to pay:

200 000 − 51282.03125

$= \frac{148717.9688}{8}$

~~R~~ 18589.74

∴ regular payment within 8 times

18589.74

M_group Solutions

Problem 1

first option (between 1 to 10 days)
if he borrows money from the bank to get ~~3%~~ 3% off of the R2800 he owes the merchant.

R 2800 − R2800 × 3% = R 2716,00
If he borrows that money from the bank
∴ he will end up paying:

$$SI = A(1 + in)$$
$$SI = R2716(1 + 1(5\%))$$
$$SI = 2851,80$$

Second option (between 10 to 30 days)
He/she will pay R 2800

Third option (pays after 1 month 30 days)
He/she will pay R2800 + R30 = R 2830

4th option (if he/she pay after 2 months)

He/she will pay R2800 + R30×2 = R2860

∴ It will be worth while or good ideas to go to the bank and borrow money to take the advantage of the discount only if he has to pay the merchant after 2 ~~years~~ months ie R8,20 more than the amount he will pay to ~~pay to~~ the bank

Problem 2

$A = P(1+i)^n$

$A = 1000\left(1 + \frac{7,65\%}{12}\right)^{18+1}$

$A = 1000\left(1 + \frac{7,65\%}{12}\right)^{19}$

$A = R1128,332125$

$Fv = x\left[\frac{(1+i)^n - 1}{i}\right]$ For monthly deposits

$Fv = 300\left[\frac{\left(1+\frac{7,65\%}{12}\right)^{18} - 1}{\frac{7,65\%}{12}}\right]$

$Fv = R\ 5702,803469$

Total Future Amount = A + Fv

= R 1128,332125 + R 5702,803469

= R 6831,135594

Money Accumulated = Total Future Amount − [Total deposits + 1000]

= 6831,135594 − [(300 x 18) + 1000]

= R 431,1355942

= R 431,14

Problem 3

$$FV = x\left(\frac{(1+i)^n - 1}{i}\right)$$

$$\frac{300000}{5000} = \frac{5000}{5000}\left[\frac{(1+i)^{35} - 1}{i}\right]$$

$$60i = (1+i)^{35} - 1$$

$$\frac{60i}{60} + \frac{1}{60} = \frac{(1+i)^{35}}{60}$$

$$i + \frac{1}{60} = \frac{(1+i)^{35}}{60}$$

$$5000 + 5000(1+i)^1 + 5000(1+i)^2 + \ldots 5000(1+i)^{?}$$

$$5000(1+i)^1 + 5000(1+i \ldots$$

$$5000 + 5000(1+i)^{35} \qquad 5000\left((1+i) + (1+i)^{34}\right.$$

$$5000\left(1 + (1+i)^{35}\right) \times 17.5$$

$$300000 = 87500\left(1 + (1+i)^{35}\right.$$

$$\frac{300000}{87500} = 1 + (1+i)^{35}$$

$$\frac{300000}{87500} - 1 = (1+i)^{35}$$

$$\left(\frac{17}{7}\right)^{1/35} = (1+i)^{35 \times \frac{1}{35}}$$

$$\left(\frac{17}{7}\right)^{1/35} = 1 + i$$

$$i = \left(\frac{17}{7}\right)^{1/35} - 1$$

$$i = 0{,}0256756025 \times 100$$

$$i = 2{,}57\%$$

Problem 4

$$CI = A[1 + i]^n$$

$$= 5000\left[1 + \frac{5\%}{2}\right]^{2\times 2}$$

$$= R220762,581$$

$$R220762,5781 - R200000$$

$$R20762,57813$$

$$CI = 20762,57813\left[1 + \frac{5\%}{2}\right]^2$$

$$= R21813,68364$$

$$PV = \frac{x\left[1 - (1+i)^{n}\right]}{i}$$

~~$= 500 \; 21813,68364\left[1 - \left(1+\frac{5\%}{2}\right)^8\right.$~~

$$\frac{21813,68364 \;\; \frac{5\%}{2}}{1 - \left(1+\frac{5\%}{2}\right)^4} = x$$

$$\therefore x = 5798,46709$$

Appendix J: Questionnaire

Research question: What are Pre-Service Mathematics Teachers espoused conceptions concerning non-routine mathematical problem solving?

Instruction:

1. **Do not write your name**
2. **Provide an explanation/a reason for each choice in each statement.**

NB:

Your responses will be treated as confidential and will only be used for research purposes only. No points/marks are earned by partaking in this exercise.

1 Is it important for learners to know many methods (ways) to solve a mathematics problem? Explain.

2 Is it good for learners to learn one method (way) of solving a mathematics problem but practice it on many problems?

3 What should be considered as a mathematical problem to learners? Give examples.

Do you agree or disagree with each statement (3-26) below? Provided reasons for your choice.

4 A vital task for the teacher is motivating children to solve their own mathematical problems.

5 Ignoring the mathematical ideas that children generate themselves can seriously limit their learning.

6 It is important for children to be given opportunities to reflect on and evaluate their own mathematical understanding.

7 It is important for teachers to understand the structured way in which mathematics concepts and skills relate to each other.

8 Knowing how to solve a mathematics problem is as important as getting the correct solution.

9 Teachers of mathematics should be fascinated with how children think and intrigued by alternative ideas.

10 Providing children with interesting problems to investigate in small groups is an effective way to teach mathematics.

11 Allowing a child to struggle with a mathematical problem, even a little tension, can be necessary for learning to occur.

12 Children always benefit by discussing their solutions to mathematical problems with each other.

13 Persistent questioning has a significant effect on children's mathematical learning.

14 Justifying the mathematical statements that a person makes is an extremely important part of mathematics.

15 As a result of my experience in mathematics classes, I have developed an attitude of inquiry.

16 Teachers can create, for all children, a non-threatening environment for learning mathematics.

17 It is the teacher's responsibility to provide children with clear and concise solution methods for mathematical problems.

18 There is an established amount of mathematical content that should be covered at each grade level.

19 It is important that mathematics content be presented to children in the correct sequence.

20 Mathematical material is best presented in an expository style: demonstrating, explaining and describing concepts and skills.

21 Mathematics is computation.

22 Telling the children the answer is an efficient way of facilitating their mathematics learning.

23 I would feel uncomfortable if a child suggested a solution to a mathematical problem that I hadn't thought of previously.

24 It is not necessary for teachers to understand the source of children's errors; follow-up instruction will correct their difficulties.

25 Listening carefully to the teacher explain a mathematics lesson is the most effective way to learn mathematics.

26 It is important to cover all the topics in the mathematics curriculum in the textbook sequence.

27 If a child's explanation of a mathematical solution doesn't make sense to the teacher it is best to ignore it.