A dissertation on

BIANCHI-I COSMOLOGICAL MODEL IN $f(R,T)$ GRAVITY

submitted by

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under the supervision of

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Declaration

I, Siwaphiwe Jokweni, assert that the the dissertation entitled **"BIANCHI – I COSMOLOGICAL MODEL IN $f(R,T)$ GRAVITY"** is submitted by the undersigned to the Department of Mathematics, University of Zululand, South Africa, for the award of Master of Science in Applied Mathematics. I solemnly declare that the work complied in this dissertation is the original work embodied from the research done by me during the period of my master program, under the supervision of Prof. Aroonkumar Beesham and Co-supervision of Dr. Vijay Singh. This research work compiled in this study has not been submitted in any form to another university for any degree. Any graphs, data, and other information that did not originate from my work are duly acknowledged.

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This is to certify that the dissertation entitled **“BIANCHI–I COSMOLOGICAL MODEL IN $f(R,T)$ GRAVITY"** is submitted by **Mr. Siwaphiwe Jokweni**, to the Department of Mathematics, University of Zululand, South Africa, for the award of Master of Science in Applied Mathematics. This is his original work, which is embodied from the research carried out by him during his postgraduate period under our supervision. To the best of our knowledge, the work reported in this dissertation has not been submitted to any other Institution or University in any form for the award of any degree or diploma.

Supervisor : **Prof. Aroonkumar Beesham**

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Co-Supervisor : **Dr. Vijay Singh**

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DEDICATED

TO

MY ENTIRE FAMILY

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Preface

Cosmology today is often described as being a 'precision' science, which reflects that cosmology had not always been seen as such. The enormous theoretical and experimental data from the cosmic microwave background (CMB), type Ia supernovae (SNe Ia), the Wilkinson Microwave Anisotropy Probe (WMAP), large scale structure (LSS), gravitational lensing, the Sloan Digital Sky Survey (SDSS), baryonic acoustic oscillations (BAO), and PLANCK, has drastically ameliorated cosmology; thus, providing a deeper understanding of the universe. These observations suggest that the universe is currently undergoing an accelerated expansion, where two thirds of its critical energy density is reserved in the form of an energy called dark energy (DE). This energy is usually associated with a cosmological constant, and the resulting standard cosmological model is called the ΛCDM model. This humongous episode confronted the fundamental theories of cosmology and astrophysics. Due to some shortcomings of the ΛCDM model, various alternatives have been proposed, which include modifications of general relativity itself, by imposing extra terms in the Einstein-Hilbert action (EH), or by considering dynamical candidates. These modified theories of gravity include Gauss-Bonnet, $f(G)$, higher derivative (HD) theories, $f(R)$ theories, $f(T)$ and $f(R,T)$ gravity theories, while dynamical candidates include the cosmological constant, quintessence, phantom, quintom, k-essence, tachyon and Chaplygin gas, among others.

Though the present universe is homogenous and isotropic, theoretical studies and observational data support the existence of an anisotropic phase at early evolution, leading to the consideration of anisotropic- background models of the universe. Many authors have explored the features of modified theories of gravity in anisotropic background to study the early universe. Amongst the various families of homogeneous, but anisotropic geometries, the most well-known are the Bianchi type I -IX space-time line elements. However, earlier studies on the possible effects of anisotropic universe make the Bianchi type-I model a prime alternative. In particular, a locally-rotationally-symmetric (LRS) or a plane symmetric spacetime is the simplest version of Bianchi-I models.

Most studies on standard gravity, as well as on modified gravity, assume the cosmic fluid to be prefect, i.e. non-viscous. From a hydrodynamicist's point of view, this is somewhat visionary, since there are several mechanisms in fluid mechanics, even in homogeneous space without boundaries, it is where bulk viscous fluid come into play. Dissipative effects, including both bulk and shear viscosity, are supposed to play a very important role in the early evolution of the universe. The bulk viscous pressure term in the matter energy-momentum tensor may lead to an accelerating universe.

This dissertation primarily investigates exact solutions of LRS Bianchi-I cosmological model with and without viscous matter in $f(R,T)$ theory of gravity, where $f(R,T)$ is an arbitrary function of the Ricci scalar R and the trace T of the energy-momentum tensor. In particular, we have studied $f(R,T) = R + 2f(T)$, where $f(T) = \lambda T$ with λ being an arbitrary constant. The function $f(R,T) = R + 2f(T)$ is used with two non-interacting fluids: one the perfect fluid, and the other from modified $f(R,T)$ gravity. The characteristic of the dynamical evolution of each cosmological model has been performed. A number of viability criteria, such as the existence of exact real solutions and physical viability, have been taken care of from each cosmological model.

This is a four-chapter dissertation comprising the first introductory chapter; chapters two and three, being the actual research work, carried out by the authors; and the concluding chapter.

Chapter 1

This introductory chapter gives the overview of the sequential understanding of the universe, from the inflationary phase, through the radiation and matter phases, and up to its current state, the accelerating epoch. Various concepts and cosmological parameters that explain both geometric and physical properties of the universe ought to be stated, including the new proposed alternatives of DE, modified theories of gravity, and all terms and equations involved in the theory of gravity.

Chapter 2

In this chapter, an LRS Bianchi type-I cosmological model is explored in the presence and absence of bulk viscosity within the framework of general relativity. The solutions are obtained by assuming that the expansion scalar is proportional to the shear scalar. To determine exact solutions, the system of equations is closed in two ways: first, by assuming a perfect fluid equation of state and then to study the behavior of the bulk viscous coefficient, and second, by considering two known bulk viscous coefficients, and then to study the normal matter. Comparison amongst the models is also made in two ways, firstly, by differentiating between the models of general relativity and $f(R,T)$ gravity, and secondly, differentiating between the models with and without viscosity. It is found that $f(R,T)$ gravity or bulk viscosity does not affect the behavior of the effective matter, which acts as a stiff fluid in all cases. The individual fluids have very rich behavior. The effect of $f(R,T)$ gravity is to diminish the effect of bulk viscosity.

Chapter 3

This chapter entails LRS Bianchi-I model in $f(R,T)$ gravity, where the matter is considered with and without bulk viscosity. The approach to find the solutions is the same as presented in chapter 2. To investigate the role of viscosity and $f(R,T)$ gravity in the evolution of the universe is our main concern in this study. To analyse the properties of primary matter and coupled matter, the equation of state will be another criterion to classify the nature of matter assuming two different forms of bulk viscous coefficient. Comparison amongst the models is also made in two ways: firstly, differentiating between the models of general relativity and $f(R,T)$ gravity, and secondly, differentiating between the models with and without viscosity.

Chapter 4

This chapter presents the final summary of the results obtained. The future perspectives of the work is also reported in this chapter.

Keywords

- General Relativity and Cosmology.
- The standard ΛCDM model.
- Cosmological Principle (Homogeneous and Isotropic space-time).
- Friedmann-Robertson-Walker (FRW) space-time.
- Dark Matter (DM).
- Dark Energy (DE).
- Bianchi-I space-time.
- Homogeneous and Anisotropic space-time.
- Bulk Viscosity.
- Modified theory of gravity.

Research questions

- Which cause can give the desired geometrical behaviour?
- In which matter will the model depict the desired evolution?
- What is the role of the $f(R,T)$ gravity theory?
- What is the influence of the use of different analytic forms on the bulk viscosity?
- How are the outcomes with viscous matter different from those without viscous matter?
- How are the outcomes in $f(R,T)$ gravity different from those in General Relativity?

Research aim and objectives

Aim

The aim of the research is to construct and explore the behaviour of the cosmological models in a locally-rotationally-symmetric anisotropic space-time in GR and in $f(R,T)$ gravity, in the presence and absence of viscous matter.

Objectives

The aim of the research is achieved by obtaining the exact solutions assuming that the expansion scalar is proportional to the shear scalar. Comparison amongst the models is made in two ways: by differentiating between the models of general relativity and $f(R,T)$ gravity, and by also differentiating between the models with and without viscosity.

Motivation for the study

Our universe, on a sufficiently large scale, is homogeneous and isotropic. However, on smaller scales, the universe is neither homogeneous nor isotropic. There are theoretical predictions that the universe, in its early stages, was also highly anisotropic. Among the simplest homogeneous and anisotropic models, which nevertheless completely describe the anisotropic effects, Bianchi type-I (B-I) models play outstanding role in explaining essential features of the universe, such as the formation of galaxies. Also in a universe filled with matter, initial anisotropy in a B-I universe quickly dies away and the universe eventually becomes isotropic. Since the present-day universe is isotropic, the prominent features of the B-I models make them prime candidates for studying the possible effects of anisotropy in early evolution of the universe. In particular, the locally-rotationally-symmetric (LRS) B-I space-time is one of the simplest models of an anisotropic universe that describe a homogenous and spatially flat universe. In the light of its importance in study of possible effects of anisotropy on the early universe on present-day observations, many researchers have studied the LRS B-I models in various contexts (see [1–3] and references therein).

While the perfect fluid satisfactorily accounts for the large scale matter distribution in the universe, the realistic scenario requires the consideration of matter other than a perfect fluid. Some observed physical phenomena, such as the large entropy per baryon and the noteworthy degree of isotropy of the cosmic background radiation, suggest dissipative effects in cosmology. Entropy- producing processes and dissipative effects play a very significant roles in the early evolution of the universe. In fluid cosmology, the simplest phenomenon associated with a non-vanishing entropy production is bulk viscosity. Bulk viscosity is the only dissipative effect that is consistent with the symmetry requirements of the homogeneous and isotropic Friedmann-Lemaitre-Robertson-Walker (FLRW) models (for more detail, see the review article by [4] and references therein).

As mentioned in the preface, the shortcomings of the ΛCDM model have motivated the search for alternatives to the fundamental theories of cosmology and astrophysics, which include modifications of general relativity itself by imposing extra terms in the Einstein-Hilbert action. The modified theories of gravity include higher derivative theories, Gauss-Bonnet $f(G)$ gravity, $f(R)$ theory, $f(T)$ and $f(R,T)$ gravity theories. In the past decade, $f(R,T)$ gravity has attracted the attention of many researchers to look at many astrophysical and cosmological phenomena in the context of this theory (see [5] for a broad list of references).

My intention in this dissertation is to study LRS Bianchi-I anisotropic model in $f(R,T)$ gravity with bulk viscous fluid. I try to find the exact solutions of the field equations by following existing approaches and examining the viability of the solutions. I thoroughly explored the physical behaviour of the model. Moreover, while going through the literature on such problems, to the best of my knowledge I came to know that the solutions of an LRS B-I model within the framework of GR and with an assumption of expansion scalar proportional to shear scalar have not been investigated by anyone. Therefore, before presenting the solutions of the $f(R,T)$ gravity model, I first found the solutions in GR with and without bulk viscous matter. This helped me to distinguish between the solutions in GR and $f(R,T)$ gravity. Consequently, we can analyse the significance of $f(R,T)$ gravity and bulk viscosity.

Literature Review

Cosmology is the scientific study of the origin, evolution and ultimate fate of the whole universe on a large scale. Over the years, various theories/models have been proposed about the origin of the universe, e.g., steady-state and big bang (BB) [6]. The latter is the leading explanation about how the universe began, at its simplest. It says that the universe came into existence at a definite moment in time, some $13,6$ billion years ago, in the form of a super hot, super dense fireball of energetic radiation known as Big Bag event.

From observations and theory, cosmology has enabled a deeper understanding of the universe. Modern cosmology emerged about 100 years ago through Einstein's theory of general relativity (theory of space-time and gravitation) in 1917 [7]. GR is a well known theory of space-time and gravitation, and is widely taken as a fundamental theory to explain geometrical properties of space-time. It is based on two fundamental postulates namely:

(a) The principle of equivalence

General Relativity states that gravity is equivalent to acceleration; therefore, gravity affects measurements of space and time.

(b) The principle of general covariance

This principle states that a physical law expressed in a generally covariant manner takes the same mathematical form in all coordinate systems, and is usually expressed in terms of tensor fields.

GR vividly describes gravity as a geometry of four-dimensional curved space-time and ameliorated the understanding of the cosmos by providing, to a good approximation, an eloquent description of space-time geometry around the sun and earth. It also describes the history and expansion of the universe, the bending of light from the stars and galaxies, which are far away, and gives the physics of black-holes.

In 1998, flummoxing results emerged from the Hubble Space Telescope, where distant supernovae suggested that the universe at present is going through an accelerating phase [8–10]. A cosmic fluid (pressureless and with pressure) obeying a perfect fluid type equation of state cannot support the acceleration. GR predicts a mysterious form of energy that permeates space and accelerates the expansion of the universe. The unknown component is popularly known as **"dark energy "** (DE) [11]. The presence of DE has not been directly detected yet. However, an array of observations, viz., the CMB anisotropy [12], baryonic acoustic oscillations (BAO) [13], gravitational lensing(GL) [14], and statistics of quasars and clusters [15], etc. are indirect evidences of its existence. Investigations suggest that DE has properties, which can be explained by a cosmological constant that corresponds to vacuum energy [16, 17].

The present universe consists of non-relativistic prefect fluid, dark matter (DM) and dark energy (DE). DE makes up about 70% of matter in the universe, while DM makes up 25% and visible matter only 5% [18–20]. In the standard model, DM (which is represented by pressure-less fluid, after cooling off rapidly as the expansion takes place) was first observed by Fritz Zwicky in the Coma cluster [21]. He observed that within a cluster, galaxies were moving with higher velocities than what the collective gravity from all the clusters of galaxies would permit. Hence, it is regarded as a missing mass and was given the name DM. It is also not seen directly, but its effect is clearly observed in rotating galaxies [22].

While the latter universe broadly contains the above mentioned ingredients, the universe is supposed to be filled mainly with imperfect fluids and electromagnetic (EM) radiation. Among the various imperfect fluids, the bulk viscous matter plays an important role in early evolution of the universe. There are several processes that generate viscous effects. Singh and Beesham [23] listed some of these principal processes. These mainly include the decoupling of neutrinos during the radiation era, and the decoupling of radiation and matter during the recombination era [24]. In an early stages of evolution of the universe, when neutrino decoupling phenomena occurs, the matter behaves like a viscous fluid [25]. Bulk viscosity is also associated with the GUT phase transition and string creation [26]. The presence of bulk viscosity inaugurates many interesting features in the dynamics of the universe. Initially, it was proposed that neutrino viscosity could smooth out initial anisotropies and resulted into the isotropic universe that we see today [27]. The presence of bulk viscosity can avert the big-bang singularity too [28]. Bulk viscosity can also explain a phenomenological process of particle creation in a strong gravitational field [26]. The back-reaction effects of string creation can be modelled by a bulk viscous fluid [26]. This has fascinated a wide scrutiny across the field of cosmology and many investigators have pondered on the effects of bulk viscosity in different contexts (see for examples [29–43] and references cited in these papers).

Because of technical reasons, most of the above referred investigations have assumed homogenous and isotropic symmetries. The observational data of Cosmic Microwave Background (CMB) [44] and Wilkinson Microwave Anisotropy Probe (WMAP) [45] admit the existence of anisotropic phase and has gained a lot of interest. It is supposed that the CMB anisotropies at small angular scales form the base for the formation of discrete structures. Theoretical arguments also support the existence of an anisotropic phase that approaches an isotropic phase in late time evolution [46]. Amongst the various families of homogeneous but anisotropic [1] geometries, the most well-known are the Bianchi type I -IX space-time line elements [47]. These homogeneous and anisotropic line elements play a significant role in describing the behaviour of the early stages of the evolution of the universe. Unlike the isotropic FRW space-time metric, Bianchi type models have different scale factors in each direction, which introduces the anisotropy in the system. The simplest example of a homogeneous and anisotropic model is the

[1] Spatial sections are direction dependent.

Bianchi type-I (B-I), which is more general than flat FRW line element.

Therefore, in the search for a realistic picture of the universe in its early stages, a large number of studies have been done in anisotropic space-times as well (see [48–63] and references therein). The general B-I spacetime models have been studied by, inter alia, [51, 58, 64–72]. More specifically, [73–75] presented some LRS B-I bulk viscosity cosmological models. Though we have mentioned the works that have been done in the framework of GR, there are many other works that have been carried out in other theories of gravitation such as Brans-Dicke theory and other scalar-tensor theories [76–82].

Contents

Chapter 1

Introduction

This introductory chapter gives a short review of the basic mathematical equations that govern the evolution of the Universe. The purpose of this chapter is basically to describe several issues relating to gravitation and cosmology; that is, problems related to the early inflation and the late-time cosmic acceleration. Some modified theories of gravity are briefly introduced to explain the dark energy and dark matter phenomena. The foundation of this chapter provides the motivation for the work carried out in this dissertation.

1.1 Space-time geometry

The space-time geometry is well articulated by a line-element that gives the space-time distance between any two nearby points. The Greek index x^i with $i = (0, 1, 2, 3)$ is used to denote the arbitrary space-time coordinates. Consequently, the four dimensional space-time coordinates are (x^0, x^1, x^2, x^3), where $x^0 = t$, x^1 ,x^2,x^3 is the time and spatial coordinates, respectively. Using the Einstein's summation convention, the line-element between two points is separated by coordinate intervals, dx^i and is given by

$$ds^2 = \sum_{i,j=0}^{3} g_{ij} dx^i dx^j, \tag{1.1.1}$$

where the coefficients g_{ij} are functions of the space-time coordinates x^i, under the restriction $g = |g_{ij}| \neq 0$. The quantities g_{ij} are components of a covariant symmetric tensor of rank two, known as the metric tensor. A line-element in (1.1.1) shows the curved geometry. According to GR, space is curved in a gravitational field and the geometry of space in the gravitational field is known as Riemannian. The contravariant metric tensor is given as

$$g^{ij} = \frac{\text{cofactor of } g_{ij} \text{ in } g}{g}. \tag{1.1.2}$$

The metric tensor g^{ij} is also a symmetric tensor of rank two.

1.2 Homogeneous and isotropic space-time

On a very large scale ($\gg 100\ Mpc$) the universe is homogeneous (has spatial translation symmetry) and isotropic (has spatial rotation symmetry). It contains gravitationally clustered matter in galaxies, as well as non-clustered energy, which is distributed uniformly and looks like a uniform density cloud of dust at gigantic scales. It appears the same in every place and in all directions. At its simplest form, the line-element of a flat homogeneous and isotropic space-time is given by

$$ds^2 = c^2 dt^2 - a^2(t) \left[dx^2 + dy^2 + dz^2 \right], \tag{1.2.1}$$

where c is the speed of light, and $a(t)$ is the scale factor associated with the expansion or contraction of the universe. Usually (1.2.1) is known as the Friedmann-Robertson-

Walker (FRW) metric. For a non-flat universe, the metric in spherical coordinates is given by

$$ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right], \tag{1.2.2}$$

where κ is a constant that describes the curvature of the spatial sections, and $(0 \le \theta \le \pi)$, $(0 \le \varphi \le 2\pi)$. It is either negative, zero, positive $(-1, 0, +1)$ for open, flat, closed universes, respectively.

1.3 Homogeneous and anisotropic space-time

The observation of the Wilkinson Microwave Anisotropy Probe (WMAP) [83] and the cosmic microwave background (CMB) [12] shows that there exists an anisotropy in the early stages of the universe. These CMB anisotropies at small angular scales are the ingredients for structure formation [84]. There exists also a theoretical argument that supports the existence of an anisotropic phase, where it approaches the isotropic phase during late times [46]. The Bianchi type I–IX space-time line-elements are well-known families of homogeneous and anisotropic geometries [85]. They have different scale factors in each direction. The simplest one is the Bianchi type-I (B–I) metric, which is more general than FRW line element. The line-element of a general B–I space-time metric is given by

$$ds^2 = c^2dt^2 - A^2(t)dx^2 - B^2(t)dy^2 - C^2(t)dz^2. \tag{1.3.1}$$

If $A = B = C$, it reduces to the FRW metric, and if $A \neq B = C$ or $A = B \neq C$, it is known as locally-rotational-symmetric (LRS) Bianchi I space-time, or if $A \neq B \neq C$, the totally anisotropic B–I space-time.

1.4 Gravitational action and Einstein's field equations

Einstein's relativistic field equations, which are equivalent to Poisson's equation of Newtonian dynamics, remain the most vital equations in explaining the relationship between matter and geometry. In the Newtonian perspective, the gravitational field equation under the presence of matter is [86]

$$\nabla^2\Phi = 4\pi G\rho, \tag{1.4.1}$$

where Φ is the gravitational potential, G is the gravitational constant and ρ is the density of matter. This equation depicts the mathematical relationship between the gravitational potential Φ in space at a point and the mass density ρ in a mathematical form at that point. The replacement of Φ by the metric tensor g_{ij} comes as a result of a non- relativistic limit g_{00}, playing the role of the gravitational potential.

The EH gravitational action is used to describe GR and results in the Einstein's relativistic field equations (EFE), built on the assumption that it is a function of the metric, connected by the Levi-Civita connection, containing a second-order derivatives of the metric. One of the easiest gravitational actions is matter fields being included resulting in

$$S = \int \left(\frac{1}{2\kappa}(R) + L_m \right) \sqrt{-g} d^4x, \tag{1.4.2}$$

where $R = g_{ij}R^{ij}$ is the Ricci scalar, and R^{ij} being the Ricci tensor, (L_m) is the Lagrangian density for any matter fields and $\kappa = \frac{8\pi G}{c^4}$. Varying (1.4.2) with respect to g_{ij} results in the EFE

$$R_{ij} - \frac{1}{2} g_{ij} R = \kappa T_{ij}, \tag{1.4.3}$$

where the matter density is also a component of a second rank EMT tensor, and the right hand side is in terms of the energy tensor T_{ij} with vanishing divergence. Then (1.4.3) govern the evolution of a universe. These equations also explain the gravitational redshift, the propagation of gravitational waves, how black holes behave, how the orbit of Mercury changes, how structures are formed in the universe from planets, stars, the clusters and super-clusters of galaxies, and any matter that is observed today.

The most general matter consistent with the assumption of the cosmological principle is a perfect fluid (frictionless continuous matter), which carries information about the energy density as well as the momentum density. The energy-momentum tensor

(EMT) or stress tensor is given by

$$T_{ij} = (\rho c^2 + p)u_i u_j + p g_{ij}, \tag{1.4.4}$$

where ρc^2 is the energy density, p is the pressure for the perfect fluid and u^μ is the four-velocity such that $u_\mu u^\mu = -1$. The matter distribution (T_{ij}) is a function of time, but not of spatial coordinates, due to the spatial homogeneity.

1.5 Friedmann equations

In the co-moving coordinates system, Alexander A. Friedmann in 1922 [87], from (1.2.2), (1.4.3) and (1.4.4), derived two independent equations

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G\rho}{3}, \tag{1.5.1}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = -\frac{4\pi G p}{c^2}, \tag{1.5.2}$$

where dots represent derivative with respect to cosmic time t. The above equations are known as Friedmann's equations. From (1.5.1), Friedmann mathematically predicted that the universe is expanding [87]. This was discovered seven years prior to Hubble's discovery of an expanding universe, that left Einstein foundering upon his thoughts of his static universe, because this was unstable, meaning that the actual universe was not static. In 1927, George Lamaitre came up with an independent derivation [88] of (1.5.1). The expansion of the universe was later confirmed by George Lamaitre in 1927 [88], while Hubble, in 1929, experimentally proved the expansion of the universe [89].

Then (1.5.2) is simplified to be

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho c^2 + 3p). \tag{1.5.3}$$

This equation is called the Raychaudhuri equation, which accounts for an accelerated or contracted expansion of the universe. In the absence of k in (1.5.3), it implies that the acceleration is independent of the spatial curvature of the universe. For a perfect fluid (radiation and matter) $\frac{\ddot{a}}{a} < 0$, implying a decelerated expansion of the universe.

1.6 Energy conservation law

The EMT (1.4.4) is conserved. Differentiating (1.4.4) with respect to cosmic time t, one obtains

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}(\dot{\rho}a^2 + 2\rho a\dot{a}). \tag{1.6.1}$$

From (1.5.3) and (1.6.1), one may easily derive

$$\dot{\rho} + 3(\rho + \frac{p}{c^2})\frac{\dot{a}}{a} = 0. \tag{1.6.2}$$

All terms here have a dimension of energy density per time, which implies that the change of energy per unit time is zero.

1.7 Einstein's modified field equations

The geometry and time evolution of the universe, as predicted by Einstein's theory, are given by what is now known as the Friedmann-Robertson-Walker (FRW) model, which describes the solutions to Einstein's field equations for a spatially homogeneous and isotropic universe in which the scale factor varies with time.

In 1917, Einstein proposed a static cosmological model based on the "cosmological principle" a generalization of the Copernican principle postulating that the homogeneity and isotropy of space in be extended to include the time dimension as well.

This solution to Einstein's field equations was first put forward by Alexander Friedmann in 1922 [87] and later independently by Georges Lamaître [88]. Robertson and Walker subsequently showed that this was the only solution to the field equations consistent with spatial homogeneity and isotropy. As we discussed above, in 1917, Einstein had put forth a theory of a static universe, a solution of the general relativity field equations that is not only homogeneous and isotropic in the three spatial dimensions, but also homogeneous in time. Given the lack of compelling observational evidence to the contrary at the time, Einstein believed that an eternal universe, in which the Copernican principle held not only in three spatial dimensions but also in time, was more elegant, and hence, more plausible. In order to satisfy the gravitational field equations, Einstein had to introduce a cosmological constant term.

The EH action with a cosmological constant (CC), Λ, is modified as [90]

$$S = \int \left[\frac{1}{2\kappa}(R - 2\Lambda) + L_m \right] \sqrt{-g} d^4x, \tag{1.7.1}$$

which results in the Einstein's modified field equations

$$R_{ij} - \frac{1}{2} g_{ij} R + \Lambda g_{ij} = \kappa T_{ij}, \tag{1.7.2}$$

where Λ is measured in inverse meter squared. In a comoving coordinate system, the above equations for the metric (1.2.2) and EMT (1.4.4) yield

$$\left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{k}{a^2} - \frac{\Lambda}{3}\right) c^2 = \frac{8\pi G\rho}{3}, \tag{1.7.3}$$

$$\frac{\ddot{a}}{a} - \frac{\Lambda}{3} = -\frac{4\pi G}{3c^2}(\rho c^2 + 3p). \tag{1.7.4}$$

A positive cosmological constant (Λ) curves space-time to counteract attractive gravitation due to matter. Einstein adjusted the CC for a static solution known as the Einstein universe. Arthur Eddington figured out the instability in the Einstein static universe [91]. Then Einstein disbanded the cosmological constant from his equations calling it “the biggest blunder life" [92]. Then for a while, the CC was forgotten, but later turned out to be a strong candidate of DE that explains the late-time accelerated expansion of the universe.

1.8 Some cosmological parameters

Let us focus on some observational and theoretical cosmological parameters that will be used in this dissertation.

1.8.1 Hubble parameter

The notion that the universe is composed of many galaxies, where each assembly is similar to a Milky Way, emerged in the decade of 1920’s. Observing a galaxy through visible wavelengths, there is a shift, which is the difference between the emitted (λ_{emitted}) by a source and received wavelengths ($\lambda_{\text{received}}$) by an observer. This change is defined through a red-shift, namely

$$z = \frac{\lambda_{recieved} - \lambda_{emitted}}{\lambda_{emitted}}. \tag{1.8.1}$$

For non-relativistic motion

$$z = \frac{\Delta\lambda}{\lambda} \approx \frac{v}{c}, \tag{1.8.2}$$

where v is the receding velocity between the source and observer. The relationship between z and distance d was discovered by Edwin Hubble in 1929, where he formulated the relation

$$z = \frac{H_0}{c} d, \tag{1.8.3}$$

where H_0 is the Hubble constant, which is the recession speed over the separation between the emitting and receiving galaxies. Using (1.8.2) and (1.8.3), the following relationship is obtained

$$v = H_0 d. \tag{1.8.4}$$

This relation in which the receding speed is proportional to the separation distance is known as the Hubble-Lemaitre law. The constant H_0 is positive in an expanding universe, also showing the rate at which the universe is expanding. The precise value for H_0 is unspecified, but recent observations from WMAP, CMB, BAO, $H(z)$ suggest that $H_0 = (69.32 \pm 0.80) km/s/Mpc$ [20]. In relation with the present $H_0 = H(t_0)$, from (1.8.4), the Hubble constant has dimensions of inverse time namely, $t_H = H_0^{-1}$, known as the Hubble time and is used to predict the age of the universe. The present age of the universe is $t_H = 13.77 GYr \; (1GYr = 10^9 \; years = 1 \; Billion \; years)$. The Hubble constant will not be constant with time, due to the matter and energy density in the universe. The gravitational attraction between matter and energy slows down the expansion, which leads to a decreasing rate of $H(t)$, implying a decelerating universe. The evolution of the universe is also traced through these theoretical and observational parameters. The Hubble parameter is defined by

$$H(t) = \frac{\dot{a}(t)}{a(t)}. \tag{1.8.5}$$

This parameterizes the rate of expanding universe, thus giving a way to link up with observations for models using a scale factor. It is taken into consideration that $H(t_0) = H_0$.

1.8.2 Critical density

The future of the universe relies on the critical density. This corresponds to the density of the expanding flat universe and is given as

$$\rho_c(t) = \frac{3H^2}{8\pi G}. \tag{1.8.6}$$

It separates the bounded and unbounded cases. The present value of $H_0 = 75\, kms^{-1}Mpc^{-1}$ results in

$$\rho_c \approx 10^{-29} gm/cm^3. \tag{1.8.7}$$

1.8.3 Density parameter

This determines the spatial geometry of the universe and is given by

$$\Omega = \frac{\rho(t)}{\rho_c(t)}. \tag{1.8.8}$$

If $\Omega = 1$, the universe is flat, $\Omega > 1$ means a closed universe, and $\Omega < 1$ corresponds to an open universe. Through the observations, the present universe is close to the spatially flat geometry, i.e., $\Omega \approx 1$.

1.8.4 Deceleration parameter

It measures the rate at which the expansion of the universe is changing with time, in terms of the scale factor. It is defined as

$$q = -\frac{a\ddot{a}}{\dot{a}^2}. \tag{1.8.9}$$

If $q < 0$, it represents an accelerating universe, and if $q > 0$, a decelerating universe.

1.9 Some kinematical parameters

The evolution of the universe is clearly understood by introducing kinematical parameters for observational interest and are vital when considering that the universe is homogeneous and isotropic.

1.9.1 Some basic mathematical formalism

Let us set the speed of light to be unity, i.e., $c = 1$. As g_{ij} is a fundamental tensor which describes the local geometry of space-time, the projection tensor is given by

$$h_{ij} = g_{ij} + u_i u_j, \tag{1.9.1}$$

where u_i is the four-velocity of the fluid. The rotational tensor

$$\omega_{ij} = h_i^\alpha h_j^\beta u_{\alpha;\beta} = \frac{1}{2}\left(u_{i;\alpha} h_j^\alpha - u_{j;\alpha} h_i^\alpha\right). \tag{1.9.2}$$

The expansion

$$\theta_{ij} = h_i^\alpha h_j^\beta u_{\alpha;\beta} = \frac{1}{2}\left(u_{i;\alpha} h_j^\alpha + u_{j;\alpha} h_i^\alpha\right), \tag{1.9.3}$$

where its trace becomes $\theta \equiv \theta_i^i = u^i_{;i}$.

The shear tensor

$$\sigma_{ij} = \theta_{ij} - \frac{1}{3} h_{ij}\theta, \tag{1.9.4}$$

satisfying $\sigma_i^i = 0$.

Lastly, the fluid velocity, which is

$$u_{i;j} = \omega_{ij} + \sigma_{ij} + \frac{1}{3} h_{ij} \theta - A_i u_j, \tag{1.9.5}$$

with A_i being the four-acceleration, given by $A_i = \dot{u}_i = u^j u_{i;j}$.

1.9.2 Expansion scalar

It is the rate of expansion of the universe and is represented by θ. It is defined in the FRW model as

$$\theta = u^i_{;i} = 3\frac{\dot{a}}{a} = 3H. \tag{1.9.6}$$

In homogeneous and anisotropic space-times defined in (1.3.1), this becomes

$$\theta = \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right). \tag{1.9.7}$$

1.9.3 Anisotropy parameter

It provides information on the universe's anisotropic behavior, or the measure of isotropy departure. It is described as

$$A_p = \frac{1}{3} \sum_{i=1}^{3} \left(\frac{H_i - H}{H} \right)^2, \tag{1.9.8}$$

where $H_i (i = 1, 2, 3)$ denotes the directional Hubble parameters in the x, y, z directions, respectively. Hence the Hubble parameter in the anisotropic universe is

$$H = \frac{\dot{a}}{a} = \frac{1}{3} \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right), \tag{1.9.9}$$

corresponding to the following directional Hubble parameters

$$H_1 = \frac{\dot{A}}{A},\ H_2 = \frac{\dot{B}}{B},\ H_3 = \frac{\dot{C}}{C}. \tag{1.9.10}$$

1.9.4 Shear scalar

The shear in an anisotropic universe is measured via observations. It plays a crucial role in studying homogeneous and anisotropic universes and is defined as

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{1}{2}\left[\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right] - \frac{\theta^2}{6}, \tag{1.9.11}$$

where σ^{ij} denotes the shear tensor and is fundamentally defined as

$$\sigma_{ij} = \frac{1}{2}(u_{i;\alpha}h^{\alpha}_{j} + u_{j;\alpha}h^{\alpha}_{i}) - \frac{1}{3}\theta h_{ij}, \tag{1.9.12}$$

where $\theta \equiv \theta^i_i = u^i{}_{;i}$.

1.10 Phases of the universe

The evolving universe is assumed to have undergone four different phases. These are pre-matter phase $(p = -\rho)$ with a density nearly equal to Planck density; the radiation epoch (very high temperature); matter-dominated epoch where the content of matter in the galaxies is being described by pressure-less gas; and the last phase is known as an accelerating phase with negative pressure. These phases are characterized by an equation of state (EoS) that is defined as

$$p = (\gamma - 1)\rho c^2, \tag{1.10.1}$$

where $0 \leq \gamma \leq 1$.

The continuity equation (1.6.2) by the use of above EoS results

$$\rho = \rho(t_0)\left[\frac{a(t_0)}{a(t)}\right]^{-3\gamma}, \tag{1.10.2}$$

where t_0 represents the present time. Using (1.10.2) in (1.5.1) for a flat universe ($k = 0$), one obtains

$$a(t) = a(t_0)\left(\frac{t}{t_0}\right)^{-\frac{2}{3\gamma}}, \text{ provided } (\gamma \neq 0). \tag{1.10.3}$$

If $\gamma = 0$, one gets

$$a(t) = a(t_0)\exp\left[\left(\frac{8\pi G\rho(t_0)}{3}\right)^{\frac{1}{2}} t\right]. \tag{1.10.4}$$

1.10.1 Inflationary epoch

Inflation is a short period of drastic exponential expansion of the early universe, at the end of which the standard BB model description is applied. This phase came out with a solution to the problems of flatness, horizon and the monopole problems in the universe [93]. Inflation smoothed out the geometry of the universe to be almost flat, thus allowing matter and density to be of order unity.

During inflationary epoch $\gamma = \frac{2}{3}$; therefore,

$$\rho \propto a^{-2}. \tag{1.10.5}$$

For a flat universe ($\kappa = 0$), the scale factor evolves linearly, i.e., $a(t) \propto t$. This is also known as marginal inflationary expansion. If $\gamma = 0$, ρ = constant, and $a(t) = e^{\sqrt{H_0 t}}$, which is exponential expansion. This process is vital in addressing the horizon, flatness, monopole and entropy problems.

(a) Horizon Problem

This is generally known as the homogeneity problem or rather, the cosmological fine-tuning problem in the BB model. It is supported by the background radiation with high isotropy, meaning uniformity in every direction. We observed in all directions a CMB spectrum of thermal black-body with a temperature of 2.725 Kelvin, varying only in one part in $100,000$ from perfect isotropy. The uniformity of temperature in every direction implies that very far opposite parts of the universe had been in thermal equilibrium in the past. Since there is no possibility for causality from two distant opposite parts of the universe, due to the fact that information travels at speeds less than or equal to the speed of light, how could the same temperature be measured in every direction?.

(b) Flatness Problem

The data obtained from the CMB reveals that the geometry of the universe is nearly flat, i.e., the density parameter (Ω) is nearly or close to 1. So far there, is no reason to why the density of the universe and the critical density are of order unity. Thus, an extreme fine-tuning of conditions was required earlier, which tends to be an unbelievable coincidence. More precisely, for a flat universe, the amount of matter is adequate to halt the expansion, but inadequate to re-collapse it. This will require a lot of fine-tuning to balance the act. This is known as the flatness problem of the standard model.

(c) Monopole Problem

The grand unified theory (GUT) of particle physics predicts a high abundance of magnetic monopoles, highly massive around $10^{16}\ GeV$, during the early universe. They are more stable and indestructible particles after being created. They have been present since antiquity. The lack of observations of these monopoles is a problem.

(d) Entropy Problem

Entropy remains constant in a given comoving volume in adiabatic expansion. An explanation for the presently observed high entropy per baryon ratio in the universe is lacking in the standard model of cosmology. If the assumption that entropy is constant in adiabatic assumption were violated at some point and entropy is boosted by a large factor, then presently, this problem would be resolved. This is a similar problem to the horizon problem.

1.10.2 Radiation era

Relativistic particles probe the expansion of the early universe, which is called the radiation-dominated era. This phase is filled with isotropic black body radiation, due to which the rate of expansion of the universe expansion is slowing down. The light elements: lithium, helium and deuterium, resulting from BB Nucleosynthesis, where nuclei other than hydrogen were formed, occur during this phase [94]. The evolution of the relativistic particles occurs when $\gamma = 4/3$, using it from an EoS gives $p = \frac{-\rho c^2}{3}$, then substituting to $d(\rho c^2 a^3) = \frac{-\rho c^2 d(a^3)}{3}$, gives

$$\rho \propto a^{-4}. \tag{1.10.6}$$

Then (1.5.1) for a flat universe ($\kappa = 0$) gives $\dot{a}^2 \propto a^{-2}$, which implies $a \propto \sqrt{t}$.

Presently, the contribution of radiation is of order 10^{-5}, which is negligible when compared to the matter. Different light nuclei were formed during radiation era. Later, as the universe expanded and cooled down, the neutral atoms were produced. The decoupling process thus allowed the matter-dominated phase.

1.10.3 Matter-dominated phase

After the radiation era ended, the matter phase began. The temperature of the universe had fallen to around $3000K$, so $v \ll c$. The universe at this epoch is assumed to be filled with dust matter, which permeates space and exerts zero pressure.

The dust matter exerts no pressure ($p = 0$), therefore, in a flat matter-dominated universe

$$\rho \propto a^{-3}. \tag{1.10.7}$$

The scale factor evolves as $a = t^{\frac{2}{3}}$. This transition period from the radiation to the matter phase occurs spontaneously, due to the fact that the matter density is inversely proportional to volume.

Other than the above discussed phases, there is also the possibility of Zel'dovich or stiff matter phase. In stiff matter phase, $\gamma = 2$, such that the energy density becomes equal to the pressure, i.e., $p = \rho$, which implies $\rho = a^{-6}$, and $a \propto t^{\frac{1}{3}}$. However in an empty universe ($p = \rho = 0$), a remains constant.

1.10.4 Present accelerating phase

Inflation was the first phase of cosmic acceleration before the radiation-dominated phase. Around early 1990s, from the theoretical point of view, it was believed that the expansion of the universe after inflation had to slow down. An extraordinary discovery by the Hubble Space Telescope (HST) [8] in 1998 of observations from far away supernovae illustrates that the expansion of the universe has actually shown an acceleration in recent times. In the past two decades, cosmology has shown immense progress from various projects. This has made cosmology to be an important branch of science to study both early and late-time evolutions of the universe. In 1998, vast developments have emerged to account for two phases of cosmic acceleration. A large number of observations, such as SNe Ia [9], CMB [12], LSS [95], BAO [13], WMAP [83] and recently one from the PLANCK Collaboration [96], confirm this.

Hence, there were suggestions from theorists to explain the late time acceleration. The most successful explanation is the cosmological constant, strange energy-fluid filling space, and the second is the extension of Einstein's theory of gravity.

As there is no bona-fide name to explain this enigma of late-time acceleration of the universe, theorists have termed it DE. It is known to be a property of space, or the hypothetical type of energy that is filling up space, thus triggering the universe to undergo the current accelerated expansion. An idea of an empty space to be filled up with its own energy was developed long ago by Einstein. As elucidated from its definition, DE appears as a result of more space. This idea has led to the understanding that this energy causes a drastic expansion of the universe. This kind of energy possesses the same characteristics of a CC, making it a prime candidate for DE [16], [17]. A model based on CC is known as ΛCDM model [97].

1.11 The ΛCDM Model

The CC is equivalent to vacuum energy [98] or DE [98], [22], which is different from DM [18–20].

If CC is taken to the right hand side of equation (1.7.2), the field equations are written as

$$R_{ij} - \frac{1}{2}g_{ij}R = \kappa(T_{ij} + T_{ij}^{vac}), \tag{1.11.1}$$

where $T_{ij}^{vac} = -\frac{\Lambda}{\kappa}g_{ij}$ is the vacuum contribution.

The energy-momentum tensor of vacuum energy is given by

$$T_{ij}^{vac} = (\rho_{vac}c^2 + p_{vac})u_i u_j + p_{vac}g_{ij}, \tag{1.11.2}$$

with $\rho_{vac}c^2 = \frac{\Lambda}{\kappa}$, and $p_{vac} = -\rho_{vac}c^2$.

This model is the standard model in cosmology, also known as the concordance model [99].

1.12 The success and shortcomings of the ΛCDM Model

The standard big-bang model of the universe remains successful in explaining the following predictions:

(i) Hubble's law must hold for the universe.

(ii) Formation of light atomic nuclei from protons and neutrons a few minutes after the initial singularity (BB). This unveils abundance ratios for ^3He, ^2H, ^4He and ^7Li.

(iii) The relic (CMB) having a black-body spectrum with a temperature of $2.75K$, presently.

Though the ΛCDM model has exceptional features, it encounters very momentous problems, such as the fine tuning problem of cosmological constant and coincidence problems [17, 100], monopole problem [101], flatness and horizon problems [17, 101], and singularity problem [27, 90]. The inflationary phase proposed by Alan Guth successfully explains the flatness and horizon problems [93]. The mysteries of DE and DM remain open challenges, but interesting issues in current research scope. The difficulties of the ΛCDM model has made cosmologists adapt to new alternatives, apart from CC.

1.13 Alternatives

Since the ΛCDM model has experienced serious problems when explaining DE, cosmologists came up with two theoretical approaches to account for the unknown energy. This led to two ways to account for DE: first being dynamical candidates for DE, and second, the modified theories of gravitation. Let us first discuss the former.

1.14 Dynamical candidates for Dark Energy

Due to the aforementioned problems (fine-tuning and coincidence problems) faced by the standard model of cosmology, many dynamical candidates for DE have been proposed [102], including a time-dependent cosmological term [103]. The primary candidate is quintessence [104, 105], which uses a scalar particle field [106]. The scalar fields are triggers of inflation [106, 107]. Being a source for primordial perturbations in the early universe, scalar fields allow the structure formation [108]. The similarity in primordial dark energy (DE) through inflation and the present DE led to inflationary models being used to account for late time acceleration, where these models use the notion of scalar fields [102, 104].

The results from various observations [109] illustrate other possibilities for strange dynamical candidates for DE known as phantom field, discovered by Caldwell [110] with negative kinetic energy [111], tachyon fields [112], k-essence [112, 113], quintom [114] and Chaplygin gas [115]. All these candidates are used to account for late time acceleration [98, 107, 116]. In our work, we shall use only scalar fields. So let us discuss this in details.

Scalar field

The quintessence is regarded as a slowly evolving cosmic scalar fields (ϕ) together with a self interacting scalar potential $V(\phi)$ [102,104]. By switching the sign for kinetic energy for the Lagrangian of the standard quintessence scalar field [112,117], one would obtain a phantom. So the matter Lagrangian for a phantom or a quintessence scalar field minimally coupled to gravity is [102]

$$\mathscr{L}_\phi = \frac{1}{2}\varepsilon\nabla^\sigma\phi\nabla_\sigma - V(\phi), \tag{1.14.1}$$

where $\varepsilon = \pm 1$ corresponds to quintessence and phantoms models, respectively. A general EH action for a minimally coupled phantom or quintessence scalar field is given by

$$S = \int \left(\frac{1}{2\kappa}R + \frac{1}{2}\varepsilon\nabla^\sigma\phi\nabla_\sigma - V(\phi)\right)\sqrt{-g}d^4x, \tag{1.14.2}$$

The energy-momentum tensor of quintessence or phantom scalar field is

$$T^\phi_{ij} = \varepsilon\nabla_i\phi\nabla_j\phi - g_{ij}\left[\frac{1}{2}\varepsilon\nabla^\sigma\phi\nabla_\sigma - V(\phi)\right], \tag{1.14.3}$$

where the pressure and density are given by

$$p_\phi = \frac{1}{2}\varepsilon\dot{\phi}^2 - V(\phi), \tag{1.14.4}$$

$$\rho_\phi = \frac{1}{2}\varepsilon\dot{\phi}^2 + V(\phi). \tag{1.14.5}$$

The equation of state (EoS) is defined as

$$\omega_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\varepsilon\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\varepsilon\dot{\phi}^2 + V(\phi)}. \tag{1.14.6}$$

In observations, the EoS for quintessence is $-1 < \omega_\phi < -\frac{1}{3}$, while $\omega_\phi = -1$ for the cosmological constant and $\omega_\phi < -1$ phantom [117]. The accelerating conditions are

$\omega < \frac{-1}{3}$, $a(t) \propto t^d$ with $d > 1$ so that $p_\phi < 0$ or $\rho_\phi \propto a^{-2}$. Since $V(\phi)$ is not known, a specific function of ϕ has to be assumed, e.g., constant potential, zero, exponential law, and power law [118], [106]. The setbacks of phantom matter are: it violates the strong energy conditions [119], has negative kinetic energy and stability problems [112], [120], and has a big rip curvature singularity [121]. This model seems less realistic to account for DE as there exist alternatives for dynamical properties that give clarity [122], [123].

1.15 Modified Theories of gravity

Modifying the theory of gravity just came into existence four years after the Einstein theory of gravity. Its unique status within gravitational theories has been questioned. The general theory of relativity was enlarged to accommodate broader aspects and other unified theories to account for how the universe is evolving. It was Weyl in 1919 [124] and Eddington in 1923 that were partakers of extending the theory [91]. In 1970, an array of modified theories of general relativity emerged, which include Eddington's theory of connections [91], Weyl's scale independent theory [124], Brans Dicke scalar-tensor theory [125], Kaluza [126], and Klein [127], and Nordtvedt [128].

Around the 20th century, a lot of caution has been entrusted to this theory as a result of great anticipation from cosmology, astrophysics and high-energy physics [129]. The possibility of replacing DM or DE on cosmological scales and galactic scales was mostly considered by researchers [130], [131]. A vast number of proposals on $f(R)$ theories [130, 132], $f(T)$ [133], brane world gravity [134], Gauss-Bonnet $f(G)$ [135], and Horava-Lifshitz gravity [136], have been used to explain conundrum of the universe. These models explain the mystery of late-time cosmic acceleration together with DM and DE. There are a number of fascinating features on modified theories [137]. They are quite auspicious alternatives to dark energy, the transition phase is well articulated by these theories, inflation to late time cosmic acceleration. In an absence of exotic matter, the phase from non-phantom to phantom and the problem of coincidence may be resolved by DE being dominant through the aid of modified gravity.

Though the work done in this dissertation is based on $f(R,T)$ gravity but it would be injustice not to discuss the history of development of this theory. So let us go back to the period 1960-1970. This is the time when the modification was done by introducing some higher order correction terms in EH action.

1.15.1 Higher derivative gravity

The EH action of general relativity is neither renormalized nor conventionally quantized. In the 1960s, it was found that renormalization at one loop requires the addition of higher-order curvature terms as was shown by Utiyama and DeWitt [138]. This discovery of Utiyama and DeWitt has enhanced studies of higher-order curvature on EH action undertaken around late 1960s [139]. Renormalization of higher order terms is possible but not unitary as was discovered by Stelle in 1977 [140]. Starobinsky replaces R with $R+\lambda R^2$ and also by the addition of non-local terms from the de Sitter, radiation to the matter phase. Hence when R^2 (squared curvature term) is added to the EH action, then we get $(L = R+\lambda R^2)$ with $\lambda > 0$, a coupling constant. This is called a higher derivative theory. The Hilbert action is given by [141]

$$S = \int \left[\frac{1}{2\kappa}(R+\lambda R^2)+L_m\right]\sqrt{-g}d^4x. \tag{1.15.1}$$

With $\lambda = 0$, the standard EH action is recovered.

Varying (1.15.1) with respect to g_{ij} results in

$$R_{ij} - \frac{1}{2}g_{ij}R+\lambda\left[2R(R_{ij}-\frac{1}{4}g_{ij}R)+2(\nabla_i\nabla_j - g_{ij}\Box)R\right] = \kappa T_{ij}, \tag{1.15.2}$$

where $\nabla_\mu \nabla^\mu$ is the d'Alembert operator.

Initially, R^2 was introduced to regularize ultraviolet divergences [138]. As time progressed, it was used in cosmology, thus resulting in bouncing models that eschew the big-bang singularity [141]. This work has attributes similar to the structure of higher derivative theory [142]. This theory is most fabulous because it can explain inflation in the absence of exotic matter [143]. This was proven in the work of Starobinsky [141]. The higher-order terms are better comprehended as an effective fluid not bound to energy conditions [144]. The higher derivative can also be the alternatives to quintessence, the cosmological constant or rather phantom in the absence of exotic matter or complicated potentials [131].

Capozziello suggested that cosmic acceleration can be clearly described by terms like $\frac{1}{R}$ because when the curvature becomes minute, they become important [131]. Experimental data has found irregularities, if gravity is modified in this manner. Since GR is a well tested and robust theory, one has to be vigilant because even small perturbations result into matter instabilities [145, 146] or disseminate ghosts [147]. As elucidated, low and high curvature must be ensured to agree with observations. The positive and negative curvature corrections will illustrate late acceleration of the universe [148] and also complies with solar system constraints [148]. After the introduction of power approach [149], it was suggested that at large scales, the negative power dominates, while the positive power dominates at small scales [150]. Inflation is explained by positive powers, while late acceleration is vividly explained by negative terms [151].

Capozziello and Francaviglia [152, 153] showed that transient matter-dominated decelerated expansion is accomplished and cosmological acceleration is driven by the smooth transition in R^m gravity for $m = 2$ higher derivative theory. Early and late evolutions have been studied thoroughly by [154].

1.15.2 The $f(R)$ Theories

The $f(R)$ theories give an alternative version of DE without invoking the conventional description of a mysterious form of energy [132, 148]. In $f(R)$ gravity, varying its metric enables the addition of a scalar degree of freedom, which results in late-time cosmic acceleration triggered by the Ricci scalar. This is termed dark gravity or rather curvature DE [130, 131]. It is the most flexible theory of DE because it also accounts for natural unification of various phases of the evolution of the universe [131, 148].

This theory replaces R (Ricci scalar) by a general function $f(R)$ [155]

$$S = \int \left[\frac{1}{2\kappa}(f(R) + L_m\right] \sqrt{-g} d^4x. \tag{1.15.3}$$

Varying (1.15.3) with respect to g_{ij} gives

$$f'(R)R_{ij} - \frac{1}{2}f(R)g_{ij} - (\nabla_i\nabla_j - g_{ij}\Box)f'(R) = \kappa T_{ij}, \tag{1.15.4}$$

where a prime denotes the derivative with respect to the argument.

The theory confines the basics of higher theories [130]. A lot of work has been conducted through $f(R)$ theories to address DE, DM, accelerating expansion [130,148], and singularity problem [156]. Some of the attempts were not viable to observations [157] or have the same features as the ΛCDM model [158]. A vast number of proposals [131,144,151] have been made that pass all local observations [131]. Most of the works have been conducted through spatially isotropic space-times [131, 144, 149, 151, 159–161]. There are observational evidences of anisotropies in early universe, therefore, many researchers have been working on anisotropic models based on this theory [162–165].

1.15.3 The $f(R,T)$ Theories of gravity

The modified theories of gravity, as elucidated earlier, were alternative theories to explain the current acceleration without invoking the cosmological constant. Various ways to deviate from general relativity make it complicated to formulate modified theories because of the property of the additive structure of the Ricci scalar and the matter Lagrangian in the Hilbert action of GR. However, there is no fundamental principle on the additive property of geometry and space. In 1984, Goenner [166] came up with an idea of non-coupling between geometry and matter. The maximal extension by explicitly coupling between geometry and matter with an arbitrary function of R and L_m was proposed in 2007 [167]. The functional $f(R,L_m)$ theories were recently extended by Harko [168]. They were regarded as $f(R,L_m)$ theories of gravity [169–171].

Poplawski [172] assumed baryonic matter and DE interaction to be a varying cosmological constant. The principle of least action on a relativistic covariant model on interacting DE was implemented by Bertolami et al. [167]. The general non-minimal coupling of geometry and matter was proposed by Harko [168].

In 2011, Harko et al. [173] formulated $f(R,T)$ gravity. The action of $f(R,T)$ gravity is given by

$$S=\int\left[\frac{1}{2\kappa}f(R,T)+L_m\right]\sqrt{-g}d^4x, \tag{1.15.5}$$

where $f(R,T)$ is a function of the Ricci scalar, R and the T trace of the EMT tensor and L_m matter Lagrangian density. L_m depends only on the metric components, and not on its derivatives.

The EMT tensor, T_{ij} is

$$T_{ij}=-\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}L_m\right)}{\delta g^{ij}}, \tag{1.15.6}$$

where $T = g^{ij}T_{ij}$.

Varying (1.15.5) with respect to the metric gives

$$f_R(R,T)R_{ij} - \frac{1}{2}f(R,T)g_{ij} + (g_{ij} - \nabla_i\nabla_j f_R(R,T)) = \kappa T_{ij} - f_T(R,T)(T_{ij} + \Theta_{ij}), \quad (1.15.7)$$

where f_R and f_T are derivatives of $f(R,T)$ w.r.t R and T, respectively ∇_i is the covariant derivative and

$$\Theta_{ij} \equiv g^{\mu\nu}\frac{\delta T_{\mu\nu}}{\delta g^{ij}}, \quad \mu,\nu = 0,1,2,3. \quad (1.15.8)$$

Substituting into above results in

$$\Theta_{ij} = -2T_{ij} + g_{ij}L_m - 2g^{\zeta\tau}\frac{\partial^2 L_m}{\partial g^{ij}\partial g^{\xi\tau}}. \quad (1.15.9)$$

It has been suggested that $f(R,T)$ depends on the source term and this source term is a function of the matter Lagrangian L_m. The choice of L_m decides the field equations of a model. For example: if $L_m = -p$, one has

$$\Theta = -2T_{ij} - pg_{ij}. \quad (1.15.10)$$

In particular, when $f(R,T) = R + 2f(T)$, the field equations (1.15.7) become

$$R_{ij} - \frac{1}{2}Rg_{ij} = T_{ij} - (T_{ij} + \Theta_{ij})f'(T) + \frac{1}{2}f(T)g_{ij}, \quad (1.15.11)$$

which by the use of (1.15.10) becomes

$$R_{ij} - \frac{1}{2}Rg_{ij} = T_{ij} + (T_{ij} + pg_{ij})f'(T) + \frac{1}{2}f(T)g_{ij}. \quad (1.15.12)$$

Refer to (1.6.2), the energy density is conserved in GR, i.e., $d(\rho V) = -pdV$, where $V(= a^3)$ is the volume and ρV is the total energy. However, $f(R,T)$ gravity has an interesting property, i.e., the non-zero four divergence of the EMT tensor. The covariant derivative of (1.15.7) yields [168]

$$\nabla^i T_{ij} = -\frac{f_T(R,T)}{1+f_T(R,T)}\left[(T_{ij}+\Theta_{ij})\nabla^i ln f_T(R,T) + \nabla^i\Theta_{ij} - \frac{1}{2}g_{ij}\nabla^i T\right]. \tag{1.15.13}$$

The extra terms are generated by the non-minimal coupling of matter and geometry.

The exotic imperfect fluids, or rather quantum effects, are what makes T to be selected as an argument to the Lagrangian. The new recipe of this theory is that the addition of new terms in the field equations take the role of a CC, which are time-dependent in the gravitational field, together with the new matter. This theory relies on a source term since it couples to matter and geometry. It also has the property that the covariant divergence of stress-energy tensor does not disappear. This has led to non-geodesic motion of test particles due to the extra force [167] that has allowed studies on the Newtonian limit together with the Mercury precession.

The most prominent feature of this theory is that the coupling of matter and geometry assures that the $f(R,T)$ theory maintains an extra acceleration. What is more interesting is that an extra acceleration generates, not only from the geometry, but also in the matter part. Due to this predicament, many researchers decided to look at it vividly as it shows good signs of explaining problems in astrophysics and cosmology as well. This theory violates the first law of black-hole thermodynamics [174]. Non-equilibrium thermodynamics, using two forms of energy-momentum of dark components, was studied by Sharif and Zubhair [175]. Homogeneous and isotropic space-time has been used in many cosmological models [176–180]. Some authors have also explored $f(R,T)$ theory in anisotropic space-time [181–184]. Many researchers have reconstructed this theory to account for early and late time evolution of the universe [174, 185]. Thus, the $f(R,T)$ theory is one of the active areas of research in cosmology and astrophysics.

1.16 Viscous fluid

From a hydrodynamical viewpoint, it is almost surprising to notice that the cosmic fluid - whether considered in the early or in the late epochs - is usually taken to be non-viscous. After all, there are two viscosity (shear and bulk) coefficients naturally occurring in general linear hydrodynamics, within the linear approximation implying that one is physically considering only first order deviations from thermal equilibrium. The shear viscosity coefficient is evidently of importance when dealing with flow near solid surfaces, but it can also be crucial under boundary-free conditions such as in isotropic turbulence. In later years, it has become more common to take into account the viscosity properties of the cosmic fluid. As a result of assumed spatial isotropy in the fluid, the shear viscosity is usually left out and any anisotropic deviations, like those encountered in the Kasner universe, become rather quickly smoothened out. Thus, only the bulk viscosity coefficient, remains in the energy-momentum tensor of the fluid. Another peculiar characteristic of bulk viscosity is that it acts as a negative energy field in an expanding universe [186]. Romano and Pavon have investigated the evolution of Bianchi Type-I universe with viscous fluid [187]. An LRS Bianchi-I with cosmic string and expansion scalar proportional to shear scalar was studied by [188, 189]. A series of papers that discussed Bianchi Type-I universe with viscous fluid has been considered by [190–192]. The effect of bulk viscosity on cosmological evolution has been investigated by a number of authors [98, 193–196]. Then, in 2014, [197] studied LRS Bianchi type-I stiff fluid inflationary universe with variable bulk viscosity.

Mahanta [198] considered an LRS Bianchi-I model with bulk viscous matter in $f(R,T)$ gravity for linear and quadratic forms of $f(R,T)$. The author assumed an expansion scalar proportional to the shear scalar to solve the field equations. However, the field equations in his work contain wrong signs. Consequently, the solutions obtained by him are mathematically incorrect. Later, Shamir [199] considered some models with the same formulation without bulk viscosity. Shamir's models for $f(R,T) = R + \lambda T$ and

$f(R,T) = R + \lambda T^2$ are a particular case of Mahanta's work, but he has not acknowledged Mahanta's work. However, the field equations in Shamir's paper are correct and the solutions discussed by him are mathematically and physically valid. Then [197] investigated a model with variable bulk viscosity in the frame work of locally rotationally symmetric (LRS) Bianchi type-I space–time. Singh and Kumar used spatially homogeneous and isotropic flat FRW metric and observed that the universe accelerates or exhibits a transition from a decelerated phase to an accelerated phase under certain constraints of η_0 and η_1 [200]. Then, [201] presented non-singular Bianchi types-I and V cosmological models, in the presence of bulk viscous fluid and within the framework of $f(R,T)$ gravity theory. In addition, [202–204] used bulk viscosity matter component in an LRS B-I model with variable deceleration parameter, which shows an acceleration of the universe. Recently, Sahoo and Reddy [205] have considered bulk viscosity in an LRS B-I model in $f(R,T)$ gravity with a special type deceleration parameter. Very recently, [206, 207] have studied the general B-I and B-V bulk viscous model in $f(R,T) = R + \lambda RT$ gravity with a hybrid expansion law of the scale factor.

Our objective in this dissertation is to reconsider the model presented by Mahanta [198], with the intention to correct the field equations and solutions, and to examine the viability of the solutions. We diligently explored the physical behaviour of the model. Moreover, while digging the past works on such formulations, we came to know that the solutions of an LRS B-I model within the framework of GR and with the same assumption of expansion scalar proportional to shear scalar do not exist in the existing literature. Therefore, before presenting the solutions of the $f(R,T)$ gravity model, we first explored the solutions in GR, with and without bulk viscosity. This helps us to distinguish between the solutions in GR and $f(R,T)$ gravity. Consequently, we could analyse the significance of $f(R,T)$ gravity and bulk viscosity. In chapter 2, we shall consider LRS Bianchi-I model with viscous matter in Einstein's gravity. LRS Bianchi-I model with viscous matter in $f(R,T) = R + 2\lambda T$ gravity will be considered in chapter 3. The conclusion and future scope of the work will be discussed in chapter 4.

Chapter 2

LRS Bianchi-I model with viscous matter in Einstein's gravity

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In this chapter, we have studied locally-rotationally-symmetric Bianchi type-I cosmological models in the presence and absence of bulk viscosity within the framework of general relativity. Solutions are obtained by assuming that the expansion scalar is proportional to the shear scalar. This assumption yields a constant value for the deceleration parameter ($q = 2$). To determine exact solutions, the system of equations is closed in two ways: one, by assuming a perfect fluid equation of state and thereafter study the behaviour of the bulk viscous coefficient, and second, by considering two known bulk viscous coefficients, and then to study the normal matter.

2.1 Introduction

As we have mentioned at the end of previous chapter, Mahanta [198] considered an LRS Bianchi-I model with bulk viscous matter in $f(R,T)$ gravity. However, the field equations in his work contain wrong signs. Consequently, the solutions obtained by him are mathematically incorrect. In this chapter, we explore LRS Bianchi-I model with bulk viscous matter in Einstein's gravity in the presence and absence of bulk viscosity. This will help us to distinguish between the solutions in GR and $f(R,T)$ gravity when we shall study this model in $f(R,T) = R + 2\lambda T$ gravity in the next chapter. By comparing these models, we shall be able to analyse the significance of $f(R,T)$ gravity and bulk viscosity.

The work is organized as follows: the field equations and solutions of a model without bulk viscosity is studied in sect. 2.2 and the bulk viscous model is investigated in sect. 2.3 and its subsections. The viability of the solutions is kept a priority in each model by imposing constraints for a physically realistic scenario. The comparison is made between the models, with and without viscosity. The results are discussed in sect. 2.4.

2.2 The model without bulk viscosity

As we mentioned in sect. 1.3, the metric (1.3.1) for $A \neq B = C$ or $A = B \neq C$, represents a spatially homogeneous and anisotropic LRS B-I space-time. Let us consider the former one

$$ds^2 = dt^2 - A^2 dx^2 - B^2(dy^2 + dz^2), \tag{2.2.1}$$

where we have assumed the velocity of light to be unity, i.e., $c = 1$.

The average scale factor and average Hubble parameter (refer (1.9.9)) for the above metric are, respectively, defined as

$$a = (AB^2)^{\frac{1}{3}}, \tag{2.2.2}$$

$$H = \frac{1}{3}\left(\frac{\dot{A}}{A}+2\frac{\dot{B}}{B}\right). \tag{2.2.3}$$

The energy-momentum tensor (1.4.4) in the system of units $c = 1$ of the matter reduces to

$$T_{ij} = (\rho + p)u_i u_j - p g_{ij}. \tag{2.2.4}$$

Similarly, the Einstein field equations (1.4.3) in the system of unit $8\pi G = 1$ are read as

$$R_{ij} - \frac{1}{2}R g_{ij} = T_{ij}. \tag{2.2.5}$$

The field equations (2.2.5) for the metric (2.2.1), with the consideration of the energy-momentum tensor (2.2.4), yield

$$\left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\dot{A}\dot{B}}{AB} = \rho, \tag{2.2.6}$$

$$\left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\ddot{B}}{B} = -p, \tag{2.2.7}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = -p. \tag{2.2.8}$$

These equations consist of four unknowns, namely, A, B, p, ρ. Therefore, in order to find exact solutions, one supplementary constraint is required.

Mahanta [198] considered the expansion scalar, $\theta(= 3H)$ to be proportional to the shear scalar[1], σ, which leads to

$$A = B^n, \tag{2.2.9}$$

[1] $\sigma^2 = \frac{1}{3}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right)^2$

where n is an arbitrary constant. From (2.2.7) and (2.2.8), by the use of (2.2.9), one gets

$$\frac{\ddot{B}}{B}+(n+1)\left(\frac{\dot{B}}{B}\right)^2=0, \tag{2.2.10}$$

which gives

$$B=\beta\left[(n+2)t+c_2\right]^{\frac{1}{n+2}}. \tag{2.2.11}$$

Consequently,

$$A=\alpha\left[(n+2)t+c_2\right]^{\frac{n}{n+2}}. \tag{2.2.12}$$

The volume scale factor turns out to be

$$V=a^3=c_1^{n+2}[(n+2)t+c_2]. \tag{2.2.13}$$

The expansion scalar and the shear scalar become

$$\theta=\frac{n+2}{(n+2)t+c_2},\ \sigma^2=\frac{1}{3}\left[\frac{n-1}{(n+2)t+c_2}\right]^2. \tag{2.2.14}$$

It is noted that isotropy condition, i.e., $\frac{\sigma^2}{\theta}\rightarrow 0$ as $t\rightarrow\infty$ is satisfied. Hence, the spatial volume at $t=0$ is zero, while expansion scalar is infinite, which suggests that universe started evolving with zero volume at $t=0$, i.e., big bang scenario. Also, it is noted that the average scale factor is zero at $t=0$, implying a point type singularity. The energy density and pressure become equal

$$\rho=p=\frac{(1+2n)}{(2+n)^2t^2}. \tag{2.2.15}$$

Hence, the effective matter behaves as stiff matter. The energy density must be positive for a realistic cosmological scenario, which is possible only for $n>-1/2$. To the best of our knowledge, these solutions are new.

In section "3" of his paper, Mahanta [198] worked out some geometrical parameters which are the volume, expansion scalar and shear scalar. All these parameters are defined in terms of the metric potentials A and B. We see that the scale factors given in (2.2.11) and (2.2.12) are identical to those of Mahanta's work, though we have obtained both in Einstein's GR. Hence, the metric potentials are independent of $f(R,T)$ gravity. Consequently, all the geometrical parameters remain independent of $f(R,T)$ gravity. Thus, the geometrical behaviour of the model remains similar to the model in GR. It is to be noted that although Mahanta obtained the mathematical expressions of these parameters, he has not discussed their interpretation. For the geometrical behavior of the model, we refer to [199].

2.3 Bulk viscous model

The energy density of bulk viscous matter remains the same but the pressure in energy-momentum tensor (2.2.4) for viscous fluid modifies as

$$\bar{p} = p'_m - \xi\theta, \tag{2.3.1}$$

where p'_m is the pressure of normal matter and ξ is the coefficient of bulk viscosity.

The field equations for a viscous model also remain identical to (2.2.6)–(2.2.8) except that p is replaced by $\bar{p}$. Accordingly, the field equations now consist of five unknowns, that is, A, B, ρ, p'_m, and ξ. Therefore, to determine the exact solutions completely, we require one more constraint other than (2.2.9). The assumption (2.2.9) again leads to the solution (2.2.15), i.e., $\rho = \bar{p}$, which is identical to the model without bulk viscosity. Hence, the bulk viscosity does not affect the behavior of effective matter and it acts as stiff matter. Further, to determine p'_m and ξ, we need one more constraint to close the system of equations (2.2.6)–(2.2.8). This can be done in two ways: first, to assume an EoS that relates ρ to p'_m, and then determine ξ; and second, to assume an explicit form for ξ and then determine $\bar{p}$. We shall follow both approaches in the following section.

2.3.1 The behaviour of bulk viscous coefficient

We assume that the normal matter follows the perfect fluid EoS

$$p'_m = \omega\rho, \tag{2.3.2}$$

where $0 \leq \omega \leq 1$ is the EoS parameter.

From (2.3.1), the expression for the coefficient of bulk viscosity is obtained as

$$\xi(t) = \frac{(2n+1)(\omega-1)}{(n+2)^2 t}. \tag{2.3.3}$$

Since we have $n > -1/2$ for the energy density to be positive, the coefficient of bulk viscosity for any kind of matter, except stiff matter ($\omega = 1$), remains positive and decreases with the evolution of the universe; for example, ultra-relativistic radiation ($\omega = 1/3$), non-relativistic dust ($\omega = 0$) or even for vacuum energy ($\omega = -1$). Also, as $\xi \to 0$ as $t \to \infty$, the effect of bulk viscosity disappears at late times. In case of stiff matter, the coefficient of bulk viscosity vanishes and the solutions obtained in (2.2.15) are recovered.

2.3.2 The behaviour of matter

By assuming a perfect fluid EoS, in sects. “3" and “4.1", Mahanta [198] merely obtained the expression for the coefficient of bulk viscosity. The author, in sect. “4.2", while considering the model $f(R,T) = \lambda R + \lambda T^2$, also considered two relations between the bulk viscous coefficient and expansion scalar to study the properties of normal matter and viscous matter. However, other than the wrong signs in the field equations, there is another flaw in this model. The author over-determined the solutions in this case. One needs two constraints to close the system but Mahanta used three, i.e., “(21)", “(61)" and the perfect fluid EoS for the normal matter, i.e., $p = \varepsilon\rho$, $0 \leq \varepsilon \leq 1$.

Regardless of this, the solutions are not valid as the sign on the right hand side of field equations is wrong. Though we are not incorporating this model in this present study, we shall implement the approach of considering both the relationship between the bulk viscous coefficient and expansion scalar to know the characteristics of bulk viscous matter. In these two cases, Mahanta assumed:

(i) the coefficient of bulk viscosity is directly proportional to a positive constant ($k > 0$), i.e., $\xi\theta = k$, and

(ii) the product of bulk viscosity coefficient and expansion scalar is directly proportional to energy density, i.e., $\xi\theta = k_1\rho$, where $k_1 > 0$ is a constant.

In what follows, we shall consider two explicit cases to study the behavior of normal matter by assuming these two relations.

Case (i) $\xi\theta = k$

In this case, the EoS of parameter of matter, $\omega' = p'_m/\rho$ gives

$$\omega' = 1 + \frac{k(2+n)^2t^2}{1+2n}. \tag{2.3.4}$$

At the origin, we have $\omega' = 1$ (stiff matter). If $k > 0$, the EoS parameter starts from $\omega' = 1$ and increases with the evolution. This case corresponds to a semi-realistic EoS $p = \varepsilon p$ $(\varepsilon \geq 1)$. Many researchers [208–210] have studied cosmological models with the semi-realistic matter in forward approaches. Though Mahanta considered $k > 0$, if $k < 0$, the EoS parameter exhibits a smooth transition from $\omega' = 1$ (stiff matter) to $\omega' \to -\infty$ (phantom matter). Thus, it describes all kinds of known matter (stiff matter, radiation, and dust) including the hypothetical form of dark energy (quintessence and phantom) and cosmological constant as well. Since the model only describes the decelerated universe, the dark energy characteristics do not, in anyway, contradict with the evolution of the universe because the normal matter is not the effective matter in this model. Indeed, we have already seen that the matter effectively acts as stiff matter.

Case (ii) $\xi\theta = k_1\rho$

The EoS parameter in this case takes a constant value

$$\omega' = 1 + k_1. \tag{2.3.5}$$

Hence, if $k_1 > 0$, in this case also, the matter follows the semi-realistic EoS. Again, if $k_1 < 0$, the model renders a variety of matter, depending on the values of k_1, e.g., $\omega' = 1/3$ (radiation) for $k_1 = -2/3$, $\omega' = 0$ (dust) for $k_1 = -1$, $\omega' = -1/3$ (quintessence) for $k_1 = -4/3$, $\omega' = -1$ (cosmological constant) for $k_1 = -2$, and $\omega' < -1$ (phantom) when $k_1 < -1$. If $k_1 = 0$, we have $\omega' = 1$ (stiff matter), which implies $\xi = 0$ as $\theta = 1/t \neq 0$. Hence, in the absence of bulk viscosity, the solutions given in (2.2.15) are recovered.

The solutions discussed in the above two cases were not reported by Mahanta [198].

2.4 Discussion

Mahanta [198] studied an LRS Bianchi-I model in $f(R,T)$ gravity with bulk viscous matter. A serious issue in his work is the incorrect signs in the field equations in all three models of $f(R,T)$ considered by him. Due to this error, the solutions presented by Mahanta are mathematically, and hence, physically invalid. However, the incorrect signs in the field equations do not affect the geometrical behaviour of the model. The geometrical parameters, namely: volume, expansion scalar, Hubble parameter, and shear scalar, are all correct mathematically. However, the author has not interpreted the behavior of these parameters. Later on, Shamir [199] also studied some models with matter without viscosity within the same formulation. The field equations in Shamir's paper are correct. He evaluated these parameters and also discussed their behaviours. To obtain the solutions in both the foresaid papers, the authors have assumed an expansion scalar proportional to the shear scalar, which returns a constant value of the deceleration parameter, $q = 2$. Hence, the model can describe only the decelerated expansion of the universe.

Noticing that the solutions have not been investigated in GR earlier, in this chapter, we have investigated the model in GR with matter, in the absence and presence of bulk viscosity. We have shown that the behaviour of geometrical parameters are independent of $f(R,T)$ gravity, and remain identical to the one in GR. In their studies, both Mahanta and Shamir have not examined the viability of their solutions. Then Mahanta in sect. "3" and "4.1" of his work, merely found the expressions of the coefficient of bulk viscosity. However, there are two ways to close the system. One is to study the bulk viscosity coefficient by assuming an EoS for the matter, and the second is the vice-versa, i.e., assume an explicit form ξ and study the nature of matter. We have followed both approaches. We have studied the behaviour of normal matter for two different forms of bulk viscosity coefficient considered by Mahanta. In contrast, by assuming the matter as perfect fluid, we have studied the behaviour of the coefficient of bulk viscosity.

The solutions have been found physically viable only for $n > -1/2$. The effective matter behaves as stiff matter. When the bulk viscosity is incorporated into this model, the bulk viscosity coefficient with perfect fluid (except for stiff matter) is found to be a decreasing function of cosmic time. In case of stiff matter, the coefficient of bulk viscosity vanishes. In the reverse approach, if we consider $\xi\theta = k$, for $k > 0$, the matter follows a semi-realistic EoS, while for $k < 0$ the EoS of matter exhibits a transition from a stage of stiff matter to phantom. With the second relation $\xi\theta = k_1\rho$, the EoS of matter becomes constant ($\omega = 1 + k_1$), which also renders semi-realistic matter for $k_1 > 0$; whereas, for $k_1 < 0$, the EoS can describe a variety of matter including radiation, dust, quintessence, phantom, and cosmological constant for different choices of k_1. For $k = 0 = k_1$, the solutions reduce to the model without viscosity.

Chapter 3

LRS Bianchi-I model with viscous matter in $f(R,T)$ gravity

In this chapter, we consider the model studied in the previous chapter in $f(R,T)$ gravity. The same approach is followed to determine the solutions. Apart from differentiating between the models, with and without viscosity, we also compare the models of general relativity and $f(R,T)$ gravity. It is found that $f(R,T)$ gravity or bulk viscosity does not affect the behaviour of the effective matter, which acts as a stiff fluid in all cases. However, the individual fluids behave differently in some cases. The effect of $f(R,T)$ gravity is to diminish the effect of bulk viscosity.

3.1 Introduction

Mahanta [198] considered an LRS Bianchi-I model with bulk viscous matter in $f(R,T)$ gravity for linear and quadratic forms of $f(R,T)$. The author assumed an expansion scalar proportional to the shear scalar to solve the field equations. However, the field equations in his work contain wrong signs. Consequently, the solutions obtained by him are mathematically incorrect. Later, [199] considered some models with the same formulation without bulk viscosity. Recently, [205] have considered bulk viscosity in an LRS B-I model in $f(R,T)$ gravity with a special type of deceleration parameter. Very recently, [207] have studied the general B-I bulk viscous model in $f(R,T) = R + \lambda RT$ gravity with a hybrid expansion law of the scale factor.

Our objective in this chapter is to reconsider the model presented by [198] with the intention to correct the field equations and solutions, and to examine the viability of the solutions. Mahanta [198] merely found the expression for the coefficient of bulk viscosity by assuming that the normal matter follows the perfect fluid EoS. However, in case of the $f(R,T) = R + 2\lambda T^2$ model, the author has also studied the behaviour of matter by considering two different forms of the bulk viscosity coefficient. This approach can also be implemented in the $f(R,T) = R + 2\lambda T$ model. We follow this approach. Therefore, our solutions are an extension of Mahanta's work. It is worthwhile to mention that though a single matter content is considered in $f(R,T)$ gravity models, but due to the coupling between the trace and the matter, some extra terms appear in the field equations. These additional terms may be treated as another form of matter, and may be called coupled matter. One must study the physical properties and contribution of this coupled matter too. Following [5] and the references mentioned therein on similar works, we also studied the nature of this coupled matter. We find the constraints for the primary matter and coupled matter to satisfy the weak energy condition (WEC), which ensures the viability of the solutions.

The chapter is organized as follows: the field equations of the LRS B-I space-time within the framework of $f(R,T) = R = 2\lambda T$ gravity and their solutions without bulk viscosity are presented in sect. 3.2 and the bulk viscous model is studied in sect. 3.3. The physical properties of the effective matter and individual fluids, using the perfect fluid EoS, are analysed in the subsections. The viability of the solutions is kept a priority in each model by imposing constraints for a physically realistic scenario. The comparison is made between the models, with and without viscosity, and between the model in GR and in $f(R,T)$ gravity. The findings are presented in the discussion sect. 3.4.

3.2 The model without viscous matter

It is vital to note that ρ and p in chapter 2 are the effective energy density and pressure, respectively, while in $f(R,T)$ gravity both the physical qualities no longer epitomize the effective energy density and pressure. Indeed, the coupling between geometry and matter in $f(R,T)$ gravity introduced some additional terms visible on the right hand side of the field equations. These terms must be treated as matter that can be called coupled matter. Therefore, to distinguish between the main matter and coupled matter, we replace p with p_m and ρ with ρ_m, which represents the primary or main matter. The notations for the energy density and pressure of the coupled matter are defined in sect. 3.2.1.

The field equations for $f(R,T) = R + 2f(T)$ is given in (1.15.12), which with the new notations can be rewritten as

$$R_{ij} - \frac{1}{2}Rg_{ij} = T_{ij} + 2\left(T_{ij} + p_m g_{ij}\right)f'(T) + f(T)g_{ij}. \tag{3.2.1}$$

For $f(T) = \lambda T$, i.e., $f(R,T) = R + 2\lambda T$, where $T = \rho_m - 3p_m$, the above equations reduce to

$$R_{ij} - \frac{1}{2}Rg_{ij} = (1+2\lambda)T_{ij} + \lambda(\rho_m - p_m)g_{ij}, \tag{3.2.2}$$

which, for the metric (2.2.1) and energy-momentum tensor (2.2.4), yield

$$\left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\dot{A}\dot{B}}{AB} = (1+3\lambda)\rho_m - \lambda p_m, \tag{3.2.3}$$

$$\left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\ddot{B}}{B} = -(1+3\lambda)p_m + \lambda\rho_m, \tag{3.2.4}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = -(1+3\lambda)p_m + \lambda\rho_m. \tag{3.2.5}$$

These are the correct field equations. We can see that the terms on the RHS of the above field equations are different from “(18)–(20)" in Ref. [198]. These equations consist of four unknowns, namely A, B, p_m, ρ_m. Therefore, in order to find exact solutions, one supplementary constraint is required. We consider the same assumption that the expansion scalar, $\theta(=3H)$ is proportional to the shear scalar, σ, and leads to the Eq. (2.2.9).

On equating (3.2.4) and (3.2.5), by the use of (2.2.9), we obtain the same as (2.2.10), which leads to the same solutions (2.2.11) and (2.2.12). Using (11) and (12) in (3.2.3) and (3.2.4), for $\lambda \neq -1/2$, we obtain

$$\rho_m = p_m = \frac{(2n+1)}{(1+2\lambda)(n+2)^2 t^2}. \tag{3.2.6}$$

This is the correct expression for the energy density and pressure, which is different from the wrong one given in Eq. “(26)" by Mahanta [198]. The primary matter acts as stiff matter. The energy density and pressure decrease with the evolution. The energy density ought to be positive for any physical viable cosmological model, and it is possible if

$$\begin{aligned} \lambda &> -1/2 \quad \text{if} \quad n > -1/2, \\ \text{or } \lambda &< -1/2 \quad \text{if} \quad n < -1/2. \end{aligned} \tag{3.2.7}$$

Thus, while the solutions in GR are valid only for $n > -1/2$, $f(R,T)$ gravity makes them valid for $n < -1/2$ also.

It is worthwhile to mention here that we have obtained expression (3.2.6) without bulk viscosity but Mahanta [198] considered the bulk viscous matter to obtain expression "(24)". It is to be noted that $\bar{P}$ in "(22)–(24)" in Mahanta's paper is just a symbol P with an overhead bar. One may readily verify that there is no use of "(14)" to get the expression "(26)" in his paper. Hence, with or without bulk viscosity, one gets the same expressions for the energy density and pressure. Thus, the energy density and pressure obtained in (3.2.6) remain independent of bulk viscosity. We shall consider the model with bulk viscosity in sect. 3.3.

3.2.1 The behaviour of coupled matter

As elucidated above, ρ_m and p_m do not represent the effective matter in this model of $f(R,T)$ gravity. The terms containing λ in (3.2.3)–(3.2.4) can be associated with the coupled matter. By separating these terms from the energy density and pressure of primary matter, these equations can be expressed as

$$\left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\dot{A}\dot{B}}{AB} = \rho_m + \rho_f, \tag{3.2.8}$$

$$\left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\ddot{B}}{B} = -(p_m + p_f), \tag{3.2.9}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = -(p_m + p_f), \tag{3.2.10}$$

where $p_f = \lambda\,(3\rho_m - p_m)$ and $p_f = \lambda\,(3p_m - \rho_m)$, respectively, represent the energy density and pressure of the coupled matter, and are obtained as

$$\rho_f = p_f = \frac{2\lambda(2n+1)}{(1+2\lambda)(n+2)^2 t^2}. \tag{3.2.11}$$

Hence, the coupling terms contribute as stiff matter. The energy density and pressure decrease with the evolution. For a physically viable model, the energy density must be positive and is corroborated under the constraints

$$\begin{aligned} -\frac{1}{2} < \lambda < 0; \quad &\text{if} \quad n < -\frac{1}{2}, \\ \text{or} \;\; \lambda < -\frac{1}{2} \;\text{or}\; \lambda > 0; \quad &\text{if} \quad n > -\frac{1}{2}. \end{aligned} \tag{3.2.12}$$

These constraints, in view of (3.2.7), agree with $\lambda > 0$ and $n > -1/2$ only. Thus, in general, $f(R,T)$ gravity makes the model physically viable for $n < -1/2$ when $\lambda < -1/2$, but if we treat the matter-geometry coupling terms as matter, then the model becomes physically viable for $\lambda > 0$ and $n > -1/2$ only.

3.3 Bulk viscous model

The gravitational field equations with bulk viscous matter remain the same as given in (3.2.3)–(3.2.4) or (3.2.8)–(3.2.9), except that the pressure, p_m modifies to

$$\bar{p}_m = p'_m - \xi\theta. \tag{3.3.1}$$

Now we shall repeat the same procedure which we have followed in sect. 2.3. First, to examine the behavior of the bulk viscosity coefficient, we consider the viscous free matter to follow the prefect fluid EoS. Second, by considering the relations of bulk viscosity assumed in cases (i) and (ii) of sect. 2.3.2, we shall study the behavior of matter.

3.3.1 The behavior of bulk viscous coefficient

Using the prefect fluid EoS $p'_m = \omega\rho_m$, where $0 \leq \omega \leq 1$, we obtain

$$\xi(t) = \frac{(1+2n)(\omega-1)}{(1+2\lambda)(2+n)^2 t}, \tag{3.3.2}$$

Since $n > -1/2$ and $\lambda > 0$ for a physically viable model, for any type of matter other than stiff fluid, ξ remains positive, which decreases with the evolution and vanishes at late times. For stiff matter ($\omega = 1$), the bulk viscosity coefficient vanishes, and the solutions reduce to viscous free matter as discussed above. Thus, the behavior of the bulk viscosity coefficient is similar to the model in GR. $f(R,T)$ gravity plays no significant role, except that a large value of λ diminishes the effect of bulk viscosity.

3.3.2 The behaviour of matter

Case (i) When $\xi\theta = k$

The EoS parameter of matter, $\omega'_m = p'_m/\rho_m$, gives

$$\omega'_m = 1 + \frac{k(n+2)^2(1+2\lambda)t^2}{1+2n}. \tag{3.3.3}$$

In view of the restrictions $n > -1/2$ and $\lambda > 0$, the above EoS parameter for $k > 0$ represents semi-realistic matter, whereas if $k < 0$, it shows a transition from $\omega'_m = 1$ to $\omega'_m \to -\infty$ as $t \to 0$, which is similar to the model in GR. Hence, this also indicates that $f(R,T)$ gravity plays no significant role in this model. However, a large value of λ makes the growth of ω'_m much faster. At the origin of evolution, $\omega'_m = 1$. If $k = 0$, the solutions reduce to the model without bulk viscosity.

Case (ii) When $\xi\theta = k_1\rho$

The EoS in this case takes a constant value

$$\omega'_m = 1 + k_1, \tag{3.3.4}$$

which is identical to (2.3.5). Hence, $f(R,T)$ gravity contributes nothing new in this case.

3.4 Discussion

In this chapter, we have reconsidered the $f(R,T) = R + 2\lambda T$ gravity model studied by Mahanta [198]. A part of our work is also an extension of Shamir's work. Both authors, in their studies, did not examine the viability of their solutions. Moreover, they did not investigate the importance of $f(R,T)$ gravity. Indeed, a comparison of the outcomes with the model in GR is required in such studies. Mahanta, in sects. "3" and "4.1" of his work, merely found the expressions of the coefficient of bulk viscosity. However, there are two ways to close the system. One is to study the bulk viscosity coefficient by assuming an EoS for the matter, and the second is the vice-versa, i.e., assume an explicit form ξ and study the nature of matter. We have followed both approaches. We have studied the behaviour of normal matter for two different forms of bulk viscosity coefficient considered by Mahanta in a model $f(R,T) = R + \lambda T^2$. In contrast, by assuming the matter as perfect fluid, we have studied the behaviour of the coefficient of bulk viscosity.

Although, Shamir presented the correct solutions, he studied only the behaviour of effective matter. However, in case of $f(R,T)$ gravity, some extra terms appeared on the right hand side of the field equations. These terms can be treated as representing matter due to the coupling between matter and geometry. Therefore, we have also reconsidered Shamir's model. We have extracted and collected the coupling terms. By considering these extra terms as coupled matter, we have studied the behaviour of coupled matter too. In this way, we look into the role of $f(R,T)$ gravity in this model. Mahanta [198] has also obtained the expressions for effective viscous matter only. It has been found that the behaviour of effective matter remains unaffected in the presence or absence of bulk viscosity.

In general, the solutions in $f(R,T)$ gravity are physically viable for $\lambda > -1/2$ and $n > -1/2$ or $\lambda < -1/2$ and $n < -1/2$. However, when the coupling terms are treated as matter, then a physically viable model is possible only for $\lambda > 0$ and $n > -1/2$. The primary matter and coupled matter act as stiff matter. Thus, the behaviour of bulk viscous model in $f(R,T)$ gravity is not much significantly different from the model in GR. The only difference is that $f(R,T) = R + 2\lambda T$ gravity for large values of λ diminishes the effect of viscous matter.

Chapter 4

Conclusion and future scope of the work

Many researchers have explored cosmological models with stiff matter in the forward approach in different contexts (see for example from [57, 211, 212] and references therein). While these works utilize simplified assumptions of the EoS of stiff matter to get exact solutions, it is a natural outcome of the present study. The stiff matter cosmological models are interesting in the sense that for such models the speed of light is equal to the speed of sound [213, 214]. A realistic example of the distribution of stiff fluid is a polytropic fluid inside a star. The existence of realistic objects in the universe makes the studies of stiff matter models prominent.

As a future scope of the work, we would like to highlight that Mahanta [198] considered three forms of $f(R,T)$, which are $f(R,T) = R+2\lambda T$, $f(R,T) = \lambda R+\lambda T$, and $f(R,T) = R+\lambda T^2$. The field equations for all the three models contain wrong signs. The first two forms are, in fact, not different as the first one is a particular case of the second. Consequently, both forms give similar results. Moreover, the model with the second form is formulated in a way that the coupling terms are treated as a variable cosmological constant, $\Lambda = (\rho - p)/2$. As we have seen, the energy density and pressure of effective matter, as well as coupled matter, become equal. As a result, Λ vanishes in this formulation and the solutions reduce to the model in GR. As a matter of course, even if one considers the correct field equations, the outcomes would be identical to the model in GR. Therefore, we have not incorporated this form explicitly in our study.

It is also worthwhile mentioning here that [198] also considered $f(R,T) = R + \lambda T^2$ gravity model with bulk viscous matter. Shamir studied this form with a matter without bulk viscosity with the correct field equations. Apart from the wrong signs in the field equations, another issue in Mahanta's model is over-determination of the solutions. It is to be noted that "(58)–(60)" have five unknowns, namely H_1, H_3, ρ, P and ξ. Therefore, only three assumptions are required to close the system, but the author uses four, that is, "(27)", "(28)", "(61 or 65)" along with the EoS $P = \varepsilon\rho$. The use of the perfect fluid EoS makes the solutions over-determined. However, we have not presented the solutions for this model here for the sake of keeping our focus on the particular form of $f(R,T) = R + 2\lambda T$. We shall consider the others models in our future work.

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Chapter 5

Appendix

LRS Bianchi I model with bulk viscosity in $f(R,T)$ gravity

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Abstract

Locally-rotationally-symmetric Bianchi type-I viscous and non viscous cosmological models are explored in general relativity (GR) and in $f(R,T)$ gravity. Solutions are obtained by assuming that the expansion scalar is proportional to the shear scalar which yields a constant value for the deceleration parameter ($q = 2$). Constraints are obtained by requiring the physical viability of the solutions. A comparison is made between the viscous and non viscous models, and between the models in GR and in $f(R,T)$ gravity. The metric potentials remain the same in GR and in $f(R,T)$ gravity. Consequently, the geometrical behavior of the $f(R,T)$ gravity models remains the same as the models in GR. It is found that $f(R,T)$ gravity or bulk viscosity does not affect the behavior of effective matter which acts as a stiff fluid in all models. The individual fluids have very rich behavior. In one of the viscous models, the matter either follows a semi-realistic EoS or exhibits a transition from stiff matter to phantom, depending on the values of the parameter. In another model, the matter describes radiation, dust, quintessence, phantom, and the cosmological constant for different values of the parameter. In general, $f(R,T)$ gravity diminishes the effect of bulk viscosity.

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1 Introduction

Our universe on a sufficiently large scale is homogeneous and isotropic. However, on smaller scales it is neither homogeneous nor isotropic. There are theoretical predictions that the early universe was also highly anisotropic which has been supported by many observations [1–7]. Among the simplest homogeneous and anisotropic models, Bianchi type-I (B-I) models play an outstanding role in understanding essential features of the early universe. Also in a universe filled with matter, the initial anisotropy in a B-I universe quickly dies away and the universe eventually becomes isotropic. Since the present-day universe is isotropic, the prominent features of the B-I models make them a prime candidate for studying the possible effects of anisotropy in the early evolution of the universe. In particular, the locally-rotationally-symmetric (LRS) B-I spacetime is one of the simplified versions of the B-I model. In light of its importance, many researchers have studied the LRS B-I models in various contexts (see [8–10] and references therein).

On the other hand, although a perfect fluid satisfactorily accounts for the large scale matter distribution in the universe, the realistic cosmological scenario requires the consideration of matter other than a perfect fluid. Some observed physical phenomena such as the large entropy per baryon and the noteworthy degree of isotropy of the cosmic background radiation, suggest dissipative effects in cosmology. Entropy producing processes and dissipative effects play a very significant role in the early evolution of the universe. In fluid cosmology, the simplest phenomenon associated with a non-vanishing entropy production is bulk viscosity (for more detail see the review article by [11] and references therein).

There are several processes which generates viscous effects (see Ref. [12] for a list of some principal processes. The presence of bulk viscosity inaugurates many interesting features in the dynamics of the universe. Initially, it was proposed that neutrino viscosity could smooth out initial anisotropies and result the isotropic universe that we see today. The presence of bulk viscosity can avert the big-bang singularity too. Bulk viscosity can also explain a phenomenological process of particle creation in a strong gravitational field. The back-reaction effects of string creation can be modeled by a bulk viscous fluid. It has attracted much interest across the field of cosmology and many investigators have pondered the effects of bulk viscosity in different contexts (see for examples [13–27] and references therein). Most of these investigations are based on isotropic cosmology. However, in the search for a realistic picture of the early universe, a large number of studies have been done in anisotropic spacetimes as well (see [28–43] and references therein). The general B-I spacetime models also have been studied by many authors [31, 38, 44, 46–52, 65]. More specifically, some authors [53–55] presented LRS B-I bulk viscous cosm/ological models.

On the other hand, the shortcomings of the ΛCDM model has confronted many authors to seek various alternatives to the fundamental theories of cosmology and astrophysics, which include modifications of general relativity itself by imposing extra terms in the Einstein-Hilbert action. The modified theories of gravity include higher derivative theories, Gauss-Bonnet $f(G)$ gravity, $f(R)$ theory, $f(T)$ and $f(R,T)$ gravity theories. In the past decade, $f(R,T)$ gravity has attracted the attention of many researchers to look at many astrophysical and cosmological phenomena in the context of this theory (see [56] for a broad list of references).

Mahanta [57] considered a bulk viscous LRS B-I model in $f(R,T)$ gravity. The author assumed an expansion scalar proportional to the shear scalar to solve the field equations. Soon after, Shamir [58] considered some models under the same formulation without bulk viscosity. Later on, Sahoo and Reddy [59] presented solutions of an LRS B-I model containing bulk viscous matter in $f(R,T)$ gravity using a special type deceleration parameter. Very recently, Yadav et al. [60] have studied the general B-I bulk viscous model in $f(R,T) = R + \lambda RT$ gravity with a hybrid expansion law of the scale factor.

Our purpose in this paper is to reconsider the LRS B-I model with bulk viscosity. We eloquently explore the behavior of the model keeping in view the physical viability of the model. Before considering the $f(R,T)$ gravity model, we first discuss the solutions in GR in the presence and absence of bulk viscosity. In this way, we distinguish the outcomes of the $f(R,T)$ gravity model with that of GR and recognize the role of $f(R,T)$ gravity and bulk viscosity.

Also, Mahanta [57] in his model of $f(R,T) = R + 2\lambda T$ merely found the expression for the coefficient of bulk viscosity. While, in case of the $f(R,T) = R + 2\lambda T^2$ model, the author also studied the behavior of matter by considering two different forms of the bulk viscosity coefficient. We implement this approach to the $f(R,T) = R + 2\lambda T$ model. Therefore, our solutions are an extension of Mahanta's work. It is worthwhile to mention that though a single matter content is considered in $f(R,T)$ gravity, due to the coupling between the trace and the matter, some extra terms appear in the field equations. We treat these additional terms as coupled matter. We study the nature of this additional matter and its contribution to the cosmic evolution.

The work is organized as follows. An LRS B-I spacetime model in the presence and absence of bulk viscosity within the framework of GR is studied in Sec. 2 and in its subsections. The $f(R,T) = R + 2\lambda T$ gravity viscous and non viscous models are explored in Sec. 3 and in its subsections. The findings are accumulated in the concluding Sec 4.

2 The model in Einstein's gravity

The spatially homogeneous and anisotropic LRS B-I space-time metric is given as

$$ds^2 = dt^2 - A^2dx^2 - B^2(dy^2 + dz^2), \tag{1}$$

where A and B are the scale factors, and are functions of the cosmic time t.

The average scale factor and average Hubble parameter, respectively, are defined as

$$a = (AB^2)^{\frac{1}{3}}, \tag{2}$$

$$H = \frac{1}{3}\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right). \tag{3}$$

where a dot represents a derivative with respect to t. We consider the energy-momentum tensor of the matter as

$$T_{ij} = (\rho + p)u_i u_j - p g_{ij}, \tag{4}$$

where ρ is the energy density and p is the thermodynamic pressure of the matter. In comoving coordinates, $u^i = \delta^i_0$, where u_i is the four-velocity of the fluid that satisfies the condition $u_i u^j = 1$.

The Einstein field equations are given by

$$R_{ij} - \frac{1}{2}R g_{ij} = T_{ij}, \tag{5}$$

where $8\pi G = 1 = c$ are assumed. The field equations (5) for the metric (1), with the consideration of the energy-momentum tensor (4), yield

$$\left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\dot{A}\dot{B}}{AB} = \rho, \tag{6}$$

$$\left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\ddot{B}}{B} = -p, \tag{7}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = -p. \tag{8}$$

These equations consist of four unknowns, namely, A, B, p, ρ. Therefore, in order to find exact solutions, one supplementary constraint is required.

Mahanta [57] considered the expansion scalar, $\theta(= 3H)$ to be proportional to the shear scalar[1], σ, which leads to

$$A = B^n, \tag{9}$$

where n is an arbitrary constant. From (7) and (8), by the use of (9), one gets

$$\frac{\ddot{B}}{B} + (n+1)\left(\frac{\dot{B}}{B}\right)^2 = 0, \tag{10}$$

which gives

$$B = \beta\left[(n+2)t + c_2\right]^{\frac{1}{n+2}}. \tag{11}$$

Consequently

$$A = \alpha\left[(n+2)t + c_2\right]^{\frac{n}{n+2}}. \tag{12}$$

The energy density and pressure become equal

$$\rho = p = \frac{(1+2n)}{(2+n)^2 t^2}. \tag{13}$$

Hence, the effective matter behaves as stiff matter. The energy density must be positive for a realistic cosmological scenario which is possible only for $n > -1/2$.

In section "3" of the paper, Mahanta [57] worked out some geometrical parameters, namely, the volume, expansion scalar and shear scalar. All these parameters are defined in terms of the metric potentials A and B. We see that the scale factors given in Eqs. (11) and (12) are identical to those of

[1] $\sigma^2 = \frac{1}{3}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right)^2$

Mahanta's work though we have obtained both in GR. In fact, LHS of the field equations (7) and (8) are same, although the right hand side (RHS) has different matter in Ref. [57], but when it is used in these two equations, the RHS is cancelled out whatever may be the matter content. Hence, the metric potentials are independent of $f(R,T)$ gravity. Consequently, all the geometrical parameters remain independent of $f(R,T)$ gravity. Thus, the geometrical behavior of the model remains similar to the model in GR. We refer to Ref. [58] for the geometrical behavior of the model.

2.1 Viscous model

The energy density of bulk viscous matter remains the same but the pressure in energy-momentum tensor (4) for viscous fluid modifies as

$$\bar{p} = p'_m - \xi\theta, \tag{14}$$

where p'_m is the pressure of matter and ξ is the coefficient of bulk viscosity.

The field equations for a viscous model remain almost similar to (6)–(8) except that the pressure p is replaced by bulk viscous pressure $\bar{p}$. Therefore, the assumption (9) again leads to the solution (13), i.e., $\rho = \bar{p}$ which is identical to the non viscous model. Hence, the bulk viscosity does not affect the behavior of effective matter and it acts as stiff matter. However, it is to be noted that the new field equations consist five unknowns, namely, A, B, ρ, p'_m, and ξ. Therefore, to determine the exact solutions completely, we require one more constraint other than (9). We have two ways: first, assuming an EoS that relates ρ to p'_m, and then determine ξ; and second, assuming an explicit form for ξ and then determine $\bar{p}$. We shall follow both approaches in the following section.

2.1.1 The behavior of bulk viscous coefficient

We assume that the matter follows the perfect fluid EoS

$$p'_m = \omega\rho, \tag{15}$$

where $0 \leq \omega \leq 1$ is the EoS parameter.

From (14), the expression for the coefficient of bulk viscosity is obtained as

$$\xi(t) = \frac{(2n+1)(\omega-1)}{(n+2)^2\, t}. \tag{16}$$

Since we have $n > -1/2$ for the energy density to be positive, the coefficient of bulk viscosity for any kind of matter except stiff matter ($\omega = 1$) remains negative and increases with the evolution of the universe, for example, ultra-relativistic radiation ($\omega = 1/3$), non-relativistic dust ($\omega = 0$) or even for vacuum energy ($\omega = -1$). Also, as $\xi \to 0$ when $t \to \infty$, the effect of bulk viscosity disappears at late times. In case of stiff matter, the coefficient of bulk viscosity vanishes and the solutions obtained in (13) are recovered.

2.1.2 The behavior of matter

By assuming a perfect fluid EoS, in Sects. "3" and "4.1", Mahanta [57] merely obtained the expression for the coefficient of bulk viscosity. However, in Sect. "4.2" while considering the model $f(R,T) = \lambda R + \lambda T^2$, the author also considered two different relations between the bulk viscous coefficient and expansion scalar to study the properties of matter and viscous fluid. However, other than the wrong signs in the field equations, there is another flaw in the model $f(R,T) = \lambda R + \lambda T^2$. The author over-determined the solutions in this case. One needs two constraints to close the system but he used three, i.e., "(21)", "(61)" and the perfect fluid EoS for the matter, i.e., $p = \epsilon\rho$, $0 \leq \epsilon \leq 1$. Regardless of over determining the solutions, the sign on the right hand side of the field equations is incorrect. Though we are not incorporating this model here, but we shall use the assumptions considered by Mahanta [57]. These assumptions are: (i) the coefficient of bulk viscosity is directly proportional to a positive constant ($k > 0$), i.e., $\xi\theta = k$, and (ii) the product of bulk viscosity coefficient and expansion scalar is directly proportional to energy density, i.e., $\xi\theta = k_1\rho$, where $k_1 > 0$ is a constant. We consider both in in following cases to examine the nature of matter.

Case (i) $\xi\theta = k$

In this case the EoS parameter, $\omega' = p'_m/\rho$ gives

$$\omega' = 1 + \frac{k(2+n)^2 t^2}{1+2n}. \tag{17}$$

At the origin we have $\omega' = 1$ (stiff matter). Mahnata considered only the case when $k > 0$. If $k > 0$, the EoS parameter starts from $\omega' = 1$ and increases with the evolution. This case corresponds to a semi-realistic EoS $p = \varepsilon p$ ($\varepsilon \geq 1$). Many researchers [61–63] have studied cosmological models with the semi-realistic matter in forward approaches. However, if $k < 0$, the EoS parameter has interesting behavior. It exhibits a smooth transition from $\omega' = 1$ (stiff matter) to $\omega' \to -\infty$ (phantom matter). Thus, it describes all kinds of known matter (stiff matter, radiation and dust) including the hypothetical form of dark energy (quintessence and phantom) and cosmological constant as well. Since the model only describes the decelerated universe, the dark energy characteristics anyway do not contradict because the matter showing this characteristic is not the effective matter in this model. Indeed we have already seen that the effective matter behaves as a stiff fluid.

Case (ii) $\xi\theta = k_1\rho$

The EoS parameter in this case takes a constant value

$$\omega' = 1 + k_1. \tag{18}$$

Hence, if $k_1 > 0$, the matter in this case also follows the semi-realistic EoS. On the other hand, if $k_1 < 0$, the model renders a variety of matter depending on the values of k_1, e.g., $\omega' = 1/3$ (radiation) for $k_1 = -2/3$, $\omega' = 0$ (dust) for $k_1 = -1$, $\omega' = -1/3$ (quintessence) for $k_1 = -4/3$, $\omega' = -1$ (cosmological constant) for $k_1 = -2$, and $\omega' < -1$ (phantom) when $k_1 < -1$. If $k_1 = 0$, we have $\omega' = 1$ (stiff matter), which implies $\xi = 0$ as $\theta = 1/t \neq 0$. Hence, in the absence of bulk viscosity, the solutions given in (13) are recovered.

3 The model in $f(R,T)$ gravity

It is vital to note that ρ and p in Sect. 2. are the effective energy density and pressure, respectively while in $f(R,T)$ gravity both the physical qualities no longer epitomize the effective energy density and pressure. Indeed the coupling between geometry and matter in $f(R,T)$ gravity adds some additional terms visible, on the RHS of the field equations. These terms must be treated as matter that can be called coupled matter. Therefore, to distinguish between the main matter and coupled matter, we replace p with p_m and ρ with ρ_m, which represents the primary or main matter. The notations for the energy density and pressure of the coupled matter are defined in Sect. 3.1.

The field equations in $f(R,T) = R + 2f(T)$ gravity with the system of units $8\pi G = 1 = c$, are obtained as

$$R_{ij} - \frac{1}{2}Rg_{ij} = T_{ij} + 2\left(T_{ij} + p_m g_{ij}\right) f'(T) + f(T)g_{ij}, \tag{19}$$

where a prime stands for a derivative with respect to T. For $f(T) = \lambda T$, i.e., $f(R,T) = R + 2\lambda T$, where $T = \rho_m - 3p_m$, (19) simplifies as

$$R_{ij} - \frac{1}{2}Rg_{ij} = (1 + 2\lambda)T_{ij} + \lambda(\rho_m - p_m)g_{ij}, \tag{20}$$

which for the metric (1) and energy-momentum tensor (4), yield

$$\left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\dot{A}\dot{B}}{AB} = (1+3\lambda)\rho_m - \lambda p_m, \tag{21}$$

$$\left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\ddot{B}}{B} = -(1+3\lambda)p_m + \lambda\rho_m, \tag{22}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = -(1+3\lambda)p_m + \lambda\rho_m. \tag{23}$$

This is the correct set of field equations. One can see that the terms on the RHS of the above field equations are different from Eqs. "(18)–(20)" in Ref. [57].

Using (11), and (12) in (21) and (22) provided $\lambda \neq -1/2$, we obtain

$$\rho_m = p_m = \frac{(2n+1)}{(1+2\lambda)(n+2)^2 t^2}. \tag{24}$$

This is the correct expression for the energy density and pressure which is different from incorrect one obtained in Eq. "(26)" by Mahanta [57]. The primary matter acts as stiff matter. The energy density and pressure decrease with the evolution. The energy density ought to be positive for any physical viable cosmological model which is possible either

$$\begin{aligned} \lambda > -1/2 \quad & \text{if} \quad n > -1/2, \\ \text{or} \ \lambda < -1/2 \quad & \text{if} \quad n < -1/2. \end{aligned} \tag{25}$$

Thus, while the solutions in GR are valid only for $n > -1/2$, $f(R,T)$ gravity makes them valid for $n < -1/2$ also.

It is worthwhile to mention here that we have obtained expression (24) without bulk viscosity but Mahanta [57] considered the bulk viscous matter to obtain expression "(24)". It is to be noted that $\bar{P}$ in Eqs. "(22)–(24)" in Mahanta's paper is just a symbol P with an overhead bar. One may readily verify that there is no use of Eq. "(14)" to calculate the expression "(26)" in his paper. Hence, for viscous or non-viscous model, one gets the same expressions of the energy density and pressure. Thus, the energy density and pressure obtained in (24) remain independent of bulk viscosity. We shall consider the bulk viscous model in Sect. 3.2.

3.1 The behavior of coupled matter

As elucidated above, ρ_m and p_m do not represent the effective matter in this model of $f(R,T)$ gravity. The terms containing λ in Eqs. (21)–(23)

can be associated with the coupled matter. By separating these terms, the equations can be expressed as

$$\left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\dot{A}\dot{B}}{AB} = \rho_m + \rho_f \tag{26}$$

$$\left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\ddot{B}}{B} = -(p_m + p_f) \tag{27}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = -(p_m + p_f), \tag{28}$$

where $p_f = \lambda\,(3\rho_m - p_m)$ and $p_f = \lambda\,(3p_m - \rho_m)$, respectively, represent the energy density and pressure of the coupled matter, and are obtained as

$$\rho_f = p_f = \frac{2\lambda(2n+1)}{(1+2\lambda)(n+2)^2 t^2}. \tag{29}$$

Hence, the coupling terms contributes as stiff matter. The energy density and pressure decrease with the evolution. For a physically viable model, the energy density must be positive which is corroborated under the constraints

$$\begin{array}{rcl} -\frac{1}{2} < \lambda < 0; & \text{if} & n < -\frac{1}{2}, \\ \text{or} \quad \lambda < -\frac{1}{2} \quad \text{or} \quad \lambda > 0; & \text{if} & n > -\frac{1}{2}. \end{array} \tag{30}$$

These constraints, in view of (25), agree with $\lambda > 0$ and $n > -1/2$ only. Thus, in general, $f(R,T)$ gravity makes the model physically viable for $n < -1/2$ when $\lambda < -1/2$, but if we treat the matter-geometry coupling terms as matter, then the model becomes physically viable only for $\lambda > 0$ and $n > -1/2$.

3.2 Bulk viscous model

The gravitational field equations with bulk viscous matter remain the same as given in (21)–(23) or (26)–(28), except that the pressure, p_m is replaced with

$$\bar{p}_m = p'_m - \xi\theta. \tag{31}$$

Now we shall repeat the same procedure which we have followed in Sec. 2.1. First, to examine the behavior of the bulk viscosity coefficient, we consider the viscous free matter to follow the prefect fluid EoS. Second, by considering the relations of bulk viscosity assumed in cases (i) and (ii) of Sect. 2.1.2, we shall study the behavior of normal matter.

3.2.1 The behavior of bulk viscous coefficient

Using the prefect fluid EoS $p'_m = \omega\rho_m$, where $0 \leq \omega \leq 1$, we obtain

$$\xi(t) = \frac{(1+2n)(\omega-1)}{(1+2\lambda)\,(2+n)^2\,t}. \tag{32}$$

Since $n > -1/2$ and $\lambda > 0$ for a physically viable model, with any kind of matter except stiff fluid, ξ remains negative which increases with the evolution and vanishes at late times. For stiff matter ($\omega = 1$), the bulk viscosity coefficient vanishes, and the solutions reduce to the non viscous model as discussed above. Thus, the behavior of the bulk viscosity coefficient is similar to the model in GR. $f(R,T) = R+2\lambda T$ gravity plays no significant role, except that a large value of λ diminishes the effect of bulk viscosity.

3.2.2 The behavior of matter

Case (i) When $\xi\theta = k$

The EoS parameter of matter, $\omega'_m = p'_m/\rho_m$, gives

$$\omega'_m = 1 + \frac{k(n+2)^2(1+2\lambda)t^2}{1+2n}. \tag{33}$$

In view of the restrictions $n > -1/2$ and $\lambda > 0$, the above EoS parameter for $k > 0$ represents semi-realistic matter, whereas for $k < 0$, it shows a transition from $\omega'_m = 1$ to $\omega'_m \to -\infty$ as $t \to 0$, which is similar to the model in GR. Hence, this also indicates that $f(R,T)$ gravity plays no significant role in this model. However, a large value of λ makes the growth of ω'_m much faster. At the origin of evolution, $\omega'_m = 1$. If $k = 0$, the solutions reduce to the model without bulk viscosity.

Case (ii) When $\xi\theta = k_1\rho$

The EoS in this case takes a constant value

$$\omega'_m = 1 + k_1, \tag{34}$$

which is identical to (18). Hence, $f(R,T)$ gravity plays no role in this case.

4 Conclusion

Mahanta [57] studied an LRS Bianchi-I model in $f(R,T)$ gravity with bulk viscous matter. The signs in the field equations in all three models of $f(R,T)$

are incorrect. This minor error makes the solutions presented mathematically, and hence physically, invalid. The positive aspect is that the incorrect signs in the field equations do not affect the metric potential. Consequently, the geometrical parameters, namely, volume, expansion scalar, Hubble parameter and shear scalar are correct mathematically. However, the author has not discussed the behavior of these parameters. Later on, Shamir [58] also studied some models without bulk viscosity under the same formulation. He has discussed the geometrical behavior of the model. To obtain the solutions, the authors have assumed an expansion scalar proportional to the shear scalar, which returns a constant value of the deceleration parameter, $q = 2$. Hence, the model can describe only the decelerated expansion of the universe.

Though the present-day universe undergoes an accelerated expansionary evolution and bulk viscosity plays a very vital role in explaining this phenomenon. However, it does not exclude the existence of a decelerating phase in the early history of our universe. Mak and Harko IJMPD 11 (2002) 447, studied a causal bulk viscous cosmological fluid for a flat constantly decelerating Bianchi type I spacetime model, and showed that this model leads to a self-consistent thermodynamic description which could describe a well-determined period of the evolution of our universe. Therefore, decelerating models have their own importance to understand the early evolution of the universe.

In this paper, we have reconsidered the $f(R,T) = R+2\lambda T$ model studied by Mahanta [57]. Indeed, a comparison of the outcomes in the modified gravity model with the outcomes of the model in GR helps to understand the role of modified gravity. So before considering the $f(R,T)$ gravity model, we have studied viscous and non viscous models in GR. A part of our work is also an extension of Shamir's work. Since Shamir has discussed the geometrical behavior, we have not repeated it here. However, we have shown that these parameters are independent of $f(R,T)$ gravity. Also, while the authors in [57,58] ignored the physical viability of the models, we have obtained the constraints for a physically realistic cosmological scenario.

Mahanta [57] in Sect. "3" and "4.1" only obtained the expressions of the coefficient of bulk viscosity. Extending his work we have also studied the behavior of normal matter for two different forms of bulk viscosity coefficient considered by him in a model $f(R,T) = R + \lambda T^2$.

The model in GR has been found physically viable only for $n > -1/2$. The effective matter behaves as stiff matter irrespective of a viscous or non viscous model. In the viscous model, the bulk viscosity coefficient with perfect fluid (except for stiff matter) is found to be negative and an increasing function of cosmic time. In the case of stiff matter, the coefficient of bulk viscosity vanishes. In the reverse approach, with the first assumption $\xi\theta = k$ for $k > 0$ the matter follows a semi-realistic EoS, while for $k < 0$ the EoS of matter exhibits a transition from a stage of stiff matter to phantom. With the second assumption $\xi\theta = k_1\rho$, the EoS of matter becomes constant ($\omega = 1+k_1$), which also renders semi-realistic matter for $k_1 > 0$, whereas for $k_1 < 0$ the EoS can describe a variety of matter including radiation, dust, quintessence, phantom, and cosmological constant for different choices of k_1. If $k = 0 = k_1$, the solutions reduce to the model without viscosity.

As far as the $f(R,T)$ gravity model is concerned, Shamir [58] has studied the behavior of effective matter only. However, in case of $f(R,T)$ gravity, some extra terms appear on the right hand side of the field equations. These terms can be treated as representing some additional matter due to the coupling between matter and geometry. Therefore, by considering matter and geometry coupling terms as coupled matter, we have examined its behavior. Since the metric potential remains identical to the model in GR, the effective matter (irrespective of viscous or non viscous models) acts as stiff matter in $f(R,T)$ gravity also.

In general, the solutions in $f(R,T)$ gravity are physically viable for $\lambda > -1/2$ and $n > -1/2$ or $\lambda < -1/2$ and $n < -1/2$. However, when the coupling terms are treated as matter then a physically viable model is possible only for $\lambda > 0$ and $n > -1/2$. The primary matter as well as coupled matter acts as stiff matter. Thus, the behavior of the bulk viscous model in $f(R,T)$ gravity is almost similar to the model in GR. The only difference is that $f(R,T) = R + 2\lambda T$ gravity for large values of λ diminishes the effect of viscous matter.

Many researchers have been explored cosmological models with stiff matter in the forward approach in different contexts (see for example from [37, 65–67] and references therein). While these works utilize simplified assumptions of the EoS of stiff matter to get exact solutions, it is a natural outcome of the present study. The stiff matter cosmological models are interesting in the sense that for such models the speed of light is equal to the speed of sound [68, 69]. A realistic example of the distribution of stiff fluid is a polytropic fluid inside a star. The existence of realistic objects in the universe makes the studies of stiff matter models prominent.

It is also worthwhile mentioning here that Mahanta [57] considered three models of $f(R,T)$, namely, $f(R,T) = R + 2\lambda T$, $f(R,T) = \lambda R + \lambda T$ and $f(R,T) = R + \lambda T^2$. The sign in field equations for all three models is incorrect. The first two forms are, in fact, not different as the first one is a particular case of the second. Consequently, both forms would produce similar results. Moreover, the second model is formulated in a way that the coupling terms are treated as a variable cosmological constant, $\Lambda = (\rho - p)/2$. As we have seen, the energy density and pressure of effective matter as well as coupled matter become equal. Resultantly, Λ vanishes in such formulation and the solutions reduce to the model in GR. Consequently, even if one considers the correct sign in the field equations, the outcomes would be identical to the model in GR. Therefore, we have not studied this form explicitly.

Finally, we would like to point out that apart from the wrong signs in the field equations, Mahanta [57] in his model $f(R,T) = R + \lambda T^2$ over determined the solutions. We see that Eqs. "(58)–(60)" have five unknowns, namely, H_1, H_3, ρ, P and ξ. Therefore, only three assumptions would be required to close the system, but he used four, namely, "(27)", "(28)", "(61 or 65)" along with the EoS $P = \epsilon\rho$. We have not considered this model in the present study for the sake of keeping our paper of mandate length. Shamir [58] has studied this form without bulk viscosity. We shall consider this model with bulk viscosity somewhere else.

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